Advanced Engineering Mathematics

Professor John C. Reichert
Texas Tech University
R.J. Marks II Class Notes
(1975)

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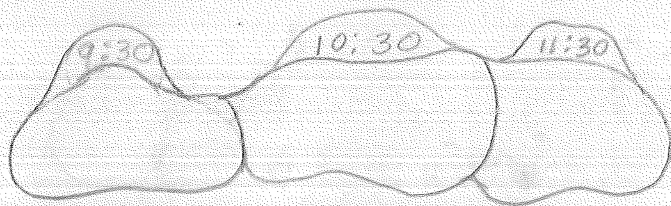


(3) $\gamma = \tan[\cos\{\sin(\gamma)\}] \Rightarrow \gamma = 0.8757798623$

(3) $\beta = \sin[\tan\{\cos(\beta)\}] \Rightarrow \beta = 0.7680431581$

(3) $\alpha = \cos[\sin\{\tan(\alpha)\}] \Rightarrow \alpha = 0.7192715209$

(2) $\psi_1 = e^{-\psi_1} \Rightarrow \psi_1 = 0.5671432904$

$\psi_2 = e^{-2\psi_2} \Rightarrow \psi_2 = 0.426302751$

(3) $\Psi = \tan e^{-\Psi} \Rightarrow \Psi = 0.6065554098$

(3) $? = e^{\tan\frac{1}{?}} \Rightarrow ? = 1.8341027773$

(3) $\phi = e^{-\tan\phi} \Rightarrow \phi = 0.5452257174$

(3) $N = e^{-\cosh N} \Rightarrow N = 0.3462649327$

(3) $\rho = \cosh e^{-\rho} \Rightarrow \rho = 1.0605551095$

(3) $q = e^{-\sinh q} \Rightarrow q = 0.5566407744$

(3) $M = \sinh e^{-M} \Rightarrow M = 0.5858351763$

(3) $\xi = \ln\cos e^{\xi} \Rightarrow \xi = -0.3023421641$

(1) $n = \cos n \Rightarrow 0.7390851332$

(2) $d = \cos\sin d \Rightarrow d = 0.7681691567$

(2) $f = \sin\cos f \Rightarrow f = 0.6948196907$

(4) $a = \ln\cos\sin\cos a = -0.3654975663$

(4) $b = \cos\ln\cos\sin b = 0.9339460396$

(4) $c = \cos\sin\cos\ln c = 0.6938513279$

(4) $d = \sin\cos\ln\cos d = 0.8039727702$

(3) $\odot = e^{\cos\sin\odot} = 1.73628491$

(3) $\otimes = \sin e^{\cos\otimes} = 0.9863379899$

(3) $\Phi = \cos\sin e^{\Phi} = 0.5517477103$

2-1. $$1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - - + + \dots$$

$$= \left(1 - \frac{1}{16} + \frac{1}{256} - \dots\right) + \left(\frac{1}{4} - \frac{1}{64} + \frac{1}{1024} - \dots\right)$$

$$= \left(1 - \frac{1}{16} + \frac{1}{256} - \dots\right) + \frac{1}{4}\left(1 - \frac{1}{16} + \frac{1}{256} - \dots\right)$$

$$= \frac{5}{4}\left(1 - \frac{1}{16} + \frac{1}{256} - \dots\right)$$

$$= \frac{5}{4}\left(1 - \left(\frac{1}{16}\right) + \left(\frac{1}{16}\right)^2 - + \dots\right)$$

$$= \frac{5}{4}\,\frac{1}{1 + 1/16} = 20/17$$

2-2. $$S = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} + \dots$$

$$f(x) = \frac{x^3}{1\cdot 3} + \frac{x^4}{2\cdot 4} + \frac{x^5}{3\cdot 5} + \frac{x^6}{4\cdot 6} + \dots$$

$$f(1) = S, \quad f(0) = 0$$

$$\frac{1}{x} f'(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

$$\frac{1}{x} f'(x)\Big|_{x=0} = 0$$

$$\frac{d}{dx}\frac{f'(x)}{x} = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$\Rightarrow \frac{f'(x)}{x} = -\log(1-x) + C_1$$

SINCE $\frac{f'(x)}{x}\Big|_{x=0} = 0, \quad C_1 = 0$

$$\Rightarrow f'(x) = -x\ln(1-x)$$

$$f(x) = -\frac{1}{2}x^2\ln(1-x) + \frac{1}{2}\ln(1-x) - (1-x) + \frac{1}{4}(1-x)^2 + C_2$$

$$f(0) = 0 \Rightarrow C_2 = 3/4$$

$$f(x) = \frac{1}{2}\,\frac{\ln(1-x)}{\frac{1}{1-x^2}} - (1-x) + \frac{1}{4}(1-x)^2 + \frac{3}{4}$$

USE LAHOPITAL'S RULE:

$$\lim_{x\to 1}\frac{\ln(1-x)}{\frac{1}{1-x^2}} = \lim_{x\to 1}\frac{\frac{-1}{1-x}}{(-1)(1-x^2)^{-2}(-2x)} = 0$$

$$\Rightarrow S = \lim_{x\to 1} f(x) = 3/4$$

2-3. $s = 1 - \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{11\cdot 3^5} + \frac{1}{13\cdot 3^6} - - + +$

$f(x) = 1 - \frac{x^5}{5\cdot 3^2} - \frac{x^7}{7\cdot 3^3} + \frac{x^{11}}{11\cdot 3^5} + \frac{x^{13}}{13\cdot 3^6} - - + +$

$f'(x) = -\frac{x^4}{3^2} - \frac{x^6}{3^3} + \frac{x^{10}}{3^5} + \frac{x^{12}}{3^6} - - + +$

$z = \frac{x^2}{3}$

$f'(x) = g(z) = -z^2 - z^3 + z^5 + z^6 - - + +$

$= (-z^2 + z^5 - z^8 + z^{11} + \dots) + (-z^3 + z^6 - z^9 + z^{12} + \dots)$

$= -z^2(1 - z^3 + z^6 - z^9 + \dots) - z^3(1 - z^3 + z^6 - z^9 + \dots)$

$= -z^2(1+z)(1 - z^3 + z^6 - z^9 + \dots)$

$= -z^2(1+z)\frac{1}{1+z^3}$

$= -\frac{\left(\frac{x^2}{3}\right)^2\left(1+\frac{x^2}{3}\right)}{1+\left(\frac{x^2}{3}\right)^3}$

$f(x) = -x + \ln\sqrt{\frac{3+3x+x^2}{3-3x+x^2}} + C$

$f(0) = 1 \Rightarrow C = 1$

$\therefore f(x) = (1-x) + \ln\sqrt{\frac{3+3x+x^2}{3-3x+x^2}}$

$s = f(1) = \ln\sqrt{7} = \frac{1}{2}\ln 7$

2-4. $S = \frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \dots$

$f(x) = \frac{x}{0!} + \frac{2x}{1!} + \frac{3x^2}{2!} + \dots$

$g(x) = \int_0^x f(x)\,dx = \frac{x}{0!} + \frac{x^2}{1!} + \frac{x^3}{2!} + \dots$

$= x\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$

$= x e^x$

$f(x) = \frac{d}{dx} g(x) = e^x + x e^x$

$S = f(1) = 2e$

2-5. $S = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots$

$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$= \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right] - \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right]$

$= \zeta(2) - \frac{1}{4}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$

$= \zeta(2) - \frac{1}{4}\zeta(2)$

$= \frac{3}{4}\zeta(2)$

$= \pi^2/8$

2-6. $S = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \dots$

$= 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$

$= \left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right] - \left[\frac{1}{2^4} + \frac{1}{4^4} + \dots\right]$

$= \zeta(4) - \frac{1}{2^4}\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right]$

$= \zeta(4) - \frac{1}{16}\zeta(4) \quad ; \zeta(4) = \frac{\pi^4}{90}$

$= \pi^4/96$

2-7. $1 - \frac{1}{4^2} + \frac{1}{9^2} - \frac{1}{16^2} + - \dots$

$= 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + - \dots$

$= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right) - \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} \dots\right)$

$= \frac{\pi^4}{96} - \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)$

$= \frac{\pi^4}{96} - \frac{1}{16}\,\zeta(4) \Leftarrow$ FROM 2-6

$= \frac{\pi^4}{96} - \frac{\pi^4}{90 \cdot 16}$

$= \pi^4 \left[\frac{15}{16 \cdot 90} - \frac{1}{16 \cdot 90}\right]$

$= \frac{14}{16 \cdot 90}\pi^4$

$= \frac{7}{8 \cdot 90}\pi^4$

$= \frac{7}{720}\pi^4$

2-8. $f(\theta) = \sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{5}\sin 3\theta + \frac{1}{7}\sin 4\theta + \ldots$

$$= \sum_1 \frac{1}{(2n-1)} \sin n\theta$$

$$= \mathrm{Im} \sum_1 \frac{1}{2n-1} e^{in\theta} = \mathrm{Im}\, g(\theta)$$

$$g(\theta) = \sum_1 \frac{1}{2n-1} e^{in\theta}$$

$$Y = e^{i\theta/2}$$

$$\Rightarrow g(Y) = \sum_1 \frac{Y^{2n}}{2n-1}$$

$$\frac{g(Y)}{Y} = \sum_1 \frac{Y^{2n-1}}{2n-1} \quad \Rightarrow \frac{g(Y)}{Y}\bigg|_{Y=0} = 0$$

$$\frac{d}{dY}\left(\frac{g(Y)}{Y}\right) = \sum_1 Y^{2n}$$

$$= \sum_0 (Y^2)^n - 1$$

$$= \frac{1}{1-Y^2} - 1$$

$$\frac{g(Y)}{Y} = \int^Y \frac{1}{1-Y^2}\,dY - Y + C$$

$$= \ln\frac{1+Y}{1-Y} - Y + C \quad \Rightarrow C = 0$$

$$= \ln\frac{1+Y}{1-Y} - Y$$

$$g(Y) = Y \ln\frac{1+Y}{1-Y} - Y^2$$

$$g(\theta) = e^{i\theta/2} \ln\frac{1+e^{i\theta/2}}{1-e^{i\theta/2}} - e^{i\theta}$$

$$\ln\frac{1+e^{i\theta/2}}{1-e^{i\theta/2}} = \ln -\left[\frac{e^{i\theta/2}+1}{e^{i\theta/2}-1}\right]$$

$$= \ln -\left[\frac{e^{i\theta/4}+e^{-i\theta/4}}{e^{i\theta/4}-e^{-i\theta/4}}\right]$$

$$= \ln -\left[\frac{1}{i\tan\theta/4}\right]$$

$$= \ln i \,\mathrm{ctn}\,\theta/4$$

$$= \ln e^{i\frac{\pi}{2}}\,\mathrm{ctn}\,\theta/4 \Leftarrow \text{ASSUMPTION}$$

$$= i\frac{\pi}{2} + \ln \mathrm{ctn}\,\theta/4$$

NOTE: $\mathrm{ctn}\frac{\theta}{4} > 0$ FOR $0 < \theta < \pi$

$\Rightarrow \ln \mathrm{ctn}\frac{\theta}{4}$ IS REAL

$$\dot{g}(\theta) = e^{i\theta/2}\left[i\,\frac{\pi}{2} + \ln \operatorname{ctn}\frac{\theta}{4}\right] - e^{i\theta}$$
$$= \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]\left[i\,\frac{\pi}{2} + \ln \operatorname{ctn}\frac{\theta}{4}\right] - e^{i\theta}$$

$$f(\theta) = \operatorname{Im} g(\theta)$$
$$= \frac{\pi}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\ln\operatorname{ctn}\frac{\theta}{4} - \sin\theta$$

$$2\text{-}9.\quad \frac{(\frac{1}{2})(\frac{1}{2})}{(9)(7)(25)(1!)} + \frac{(\frac{1}{2})(\frac{3}{2})(\frac{3}{2})}{(11)(9)(49)(2!)} + \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{5}{2})}{(13)(11)(81)(3!)} + \dots$$

$n=1 \qquad n=2 \qquad n=3$

$$\frac{(1)(1)}{2^2(9)(7)5^2(1!)} + \frac{(1)(3)(3)}{2^3(11)(9)7^2\,2!} + \frac{(1)(3)(5)(5)}{2^4(13)\cdot(11)\cdot 9^2\, 3!} + \dots$$

$$S = \sum_{n=1}^{\infty} \frac{(2n+1)(2n-1)!!}{2^{n+1}(2n+7)(2n+5)(2n+3)^2\, n!}$$

$$(2m+1)!! = \frac{(2m+1)!}{2^m\, m!}$$

$$m = n-1$$

$$\Rightarrow (2n-1)!! = \frac{(2n-1)!}{2^{n-1}(n-1)!}$$

$$C_n = \frac{(2n-1)(2n-1)!}{2^{2n}(n-1)!(2n+7)(2n+5)(2n+3)^2\, n!}$$

$$C_{n+1} = \frac{(2n+1)(2n+1)!}{2^{2n+2}(n!)(2n+9)(2n+7)(2n+5)^2(n+1)!}$$

$$= \frac{(2n+1)(2n+1)(2n)}{4\cdot n\cdot(2n+9)(2n+5)(n+1)} \cdot \frac{(2n-1)!}{2^{2n}(n-1)!(2n+7)(2n+5)}$$

?

$$= \frac{(2n+1)^2(2n)}{4n(2n+9)(2n+5)(n+1)} \cdot C_n \cdot \frac{(2n+3)^2}{(2n-1)}$$

$$\frac{C_{n+1}}{C_n} = \frac{(2n+1)^2(2n+3)^2}{2(2n+9)(2n+5)(n+1)(2n-1)}$$

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{(2n)^2(2n)^2}{2(2n)(2n)(n)(2n)} = 1$$

$$\frac{c_{n+1}}{c_n} = \frac{(4n^2+4n+1)(4n^2+12n+9)}{2(4n^2+23n+45)(2n^2+n-1)}$$

$$
\begin{array}{r}
4n^2 + 12n + 9 \\
\underline{4n^2 + 4n + 1} \\
4n^2 \quad 12n \quad 9 \\
16n^3 \quad 48n^2 \quad 36n \\
\underline{16n^4 \quad 48n^3 \quad 36n^2} \\
16n^4 + 64n^3 + 88n^2 + 48n + 9
\end{array}
$$

$$
\begin{array}{r}
4n^2 + 23n + 45 \\
\underline{2n^2 + n - 1} \\
-4n^2 \ -23n \ -45 \\
4n^3 \quad 23n^2 \quad 45n \\
\underline{8n^4 \quad 46n^3 \quad 90n^2} \\
8n^4 + 50n^3 + 109n^2 + 22n - 45
\end{array}
$$

$$\frac{9}{4}$$
$$\parallel$$
$$\begin{array}{r|l} & 1 - \frac{36}{16n} - \frac{380}{4 \cdot 16} \\ \hline 16n^4 + 100n^3 + 218n^2 + 44n - 90 & 16n^4 + 64n^3 + 88n^2 + 48n + 9 \\ & \underline{16n^4 + 100n^3 + 218n^2 + 44n - 90} \\ & -36n^3 + 130n^2 \; 4n - 81 \\ & \underline{-36n^3 + \frac{900}{4}n^2 + \frac{9 \cdot 218}{4}n - \frac{81 \cdot 9}{4}} \\ & -\frac{380}{4}n^2 + \ldots \end{array}$$

$$\frac{c_{n+1}}{c_n} = 1 - \frac{9}{4n} - \frac{95}{12n^2} + \ldots$$

$$\frac{c'_{n+1}}{c'_n} = 1 - \frac{1}{n} - \frac{s}{n \ln n} + \ldots$$

DIVERGES FOR $s > 1$

$\Rightarrow$ SERIES CONVERGES SINCE $\frac{9}{4} > 1$

$= 0.0004377$

2.10. $S = \frac{1 \cdot 3^2}{(1)(1)(1)^2} + \frac{(1 \cdot 3 \cdot 5)^2}{4 \cdot 2 \cdot (1 \cdot 2)^2} + \frac{(1 \cdot 3 \cdot 5 \cdot 7)^2}{16 \cdot 3 \cdot (1 \cdot 2 \cdot 3)^2} + \ldots$

$$C_n = \frac{[(2n+1)!!]^2}{(n!)^2 \; n \; 4^{n-1}}$$

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!}$$

$$C_n = \frac{[(2n+1)!]^2}{4^{2n+1}(n!)^4 \, n}$$

$$C_{n+1} = \frac{[(2n+3)!]^2}{4^{2n+1}[(n+1)!]^4 (n+1)}$$

$$= \frac{[(2n+3)(2n+2)(2n+1)!]^2}{4^{2n+1}[(n+1)n!]^4 (n+1)}$$

$$= \frac{(2n+3)^2(2n+2)^2}{4^2(n+1)^5} \quad \frac{[(2n+1)!]^2}{4^{2n-1}(n!)^4}$$

$$= \frac{(2n+3)^2(2n+2)^2}{16(n+1)^5} \cdot C_n \times n$$

$$\frac{C_{n+1}}{C_n} = \frac{n(2n+3)^2(2n+2)^2}{16(n+1)^5}$$

$$= \frac{4n(n+1)^2(2n+3)^2}{16(n+1)^5}$$

$$= \frac{n(2n+3)^2}{4(n+1)^3} \Rightarrow \lim_{n \to \infty} \frac{C_{n+1}}{C_n} = 1$$

$$= \frac{n(4n^2+12n+9)}{4(n^3+3n^2+3n+1)}$$

$$= \frac{4n^3+48n^2+9n}{4n^3+12n^2+12n+4}$$

$$\begin{array}{r|l} & 1 + \frac{9}{n} + \cdots \\ \hline 4n^3+12n^2+12n+4 & 4n^3+48n^2+9n \\ & \underline{4n^3+12n^2+12n+4} \\ & \qquad 36n^2-3n-4 \\ & \qquad \underline{36n^2-108n+108+\frac{36}{n}} \\ & \qquad\qquad 105n-112-\frac{36}{n} \end{array}$$

$$\frac{C_{n+1}}{C_n} = 1 + \frac{9}{n} + \ldots \Leftarrow \text{DIVERGES}$$

2-9. $f(x) = \sum_{1} \frac{(-)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = \sum_0 \frac{(-1)^n (n+1)^2 x^{2n+1}}{(2n+1)!}$

$\sin x = \sum_0 \frac{(-)^n x^{2n+1}}{(2n+1)!}$

$\frac{\sin x}{x} = \sum_0 \frac{(-)^n x^{2n}}{(2n+1)!}$ $\qquad$ $\frac{f(x)}{x} = \sum_0 \frac{(-)^n (n+1)^2 x^{2n}}{(2n+1)!}$

$z = x^2$

$\frac{\sin x}{x} = \sum_0 \frac{(-)^n z^n}{(2n+1)!}$ $\qquad$ $\frac{f(x)}{x} = \sum_0 \frac{(-)^n (n+1) z^n}{(2n+1)!}$

$= \sum_0 b_n z^n = g(z)$ $\qquad$ $= \sum_0 c_n b_n z^n$

$c_n = (n+1)^2$ $\qquad$ $g(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$

1			
	3		
4		2	
	5		0
9		2	
	7		0
16		2	
	9		0

$\Rightarrow \frac{f(x)}{x} = g(z) + 3z\, g'(z) + \frac{2z^2}{2!} g''(z) + 0$

$= \frac{\sin\sqrt{z}}{\sqrt{z}} + 3z \frac{d}{dz}\left(\frac{\sin\sqrt{z}}{\sqrt{z}}\right) + z^2 \frac{d^2}{dz^2} \frac{\sin\sqrt{z}}{\sqrt{z}}$

$\sqrt{z} = x$

$f(x) = \frac{1}{4}\sin x + \frac{3}{4} x \cos x - \frac{x^2}{4} \sin x$

2-12. $\sec z = \frac{1}{\cos z} = \sum_{n=0}^{\infty} (-)^n \frac{E_{2n}}{(2n)!} z^{2n}$

$1 = \sec z \cos z \quad ; \cos z = \sum_0 \frac{(-1)^m z^{2m}}{(2m)!}$

$\Rightarrow \sum_0 (-)^n \frac{E_{2n}}{(2n)!} z^{2n} \sum_0 \frac{(-)^m z^{2m}}{(2m)!} = \underline{1}$

$= \sum_0 \sum_0 \frac{(-z^2)^{m+n} E_{2n}}{(2m)!(2n)!} = \underline{1}$

$m = n = 0 \Rightarrow E_0 = 1$

$$\sum_{\substack{0,0 \\ n+m \neq 0}} \frac{E_{2n} (-z^2)^{m+n}}{(2m)!(2n)!} = 0$$

TRUE FOR ANY z

$$\cancel{\Rightarrow \sum_{\substack{0,0 \\ n+m \neq 0}} \frac{E_{2n}}{(2m)!(2n)!}} = \cancel{\sum_{K=0} \frac{E_n \, \delta_{K,m+n}}{(2m)!(2n)!}}$$

CONSIDER K^{TH} TERM (ie $m+n=k$)
IT MUST EQUAL ZERO

$$0 = \sum_{n+m=K} \frac{E_{2n} \, \delta_{K,m+n}}{(2m)!(2n)!}$$

NOW; IF ~~$v = n+m$~~

$$\cancel{\sum_{n=0} \sum_{m=0} f(m,n) \, \delta_{K,m+n}} = \cancel{\sum_{n=0}^{\infty} \sum_{v=n}^{\infty} f(v-n,n) \, \delta_{K,v}}$$

$$= \cancel{\sum_{n=0}^{K} f(K-n,n)}$$

$m = k - n$

$$\Rightarrow 0 = \sum_{n+m=k} \frac{E_{2n}\ \delta_{k,k}}{[2(k-n)]!\ (2n)!}$$

$$= \sum_{n=0}^{k(\infty)} \frac{E_{2n}}{[2(k-n)]!\ (2n)!}$$

$$= \sum_{n=0} \frac{E_{2n}}{(2k-2n)!\ (2n!)}$$

$$= \frac{1}{(2k)!} \sum_{n=0}^{\infty} \frac{(2k)!\ \ E_{2n}}{(2n)!\ (2k-2n)!}$$

$$= \sum_{n=0}^{\infty} \binom{2k}{2n} E_{2n}$$

$$= \sum_{n=0}^{\infty} \underbrace{[1 + (-)^n]}_{0 \text{ FOR ODD } n} \binom{2k}{n} E_n$$

$$\Rightarrow 0 = \sum_n \binom{2k}{n} E_n + \sum_n (-)^n \binom{2k}{n} E_n$$

$$= (1+E)^{2k} + (1-E)^{2k} \quad ; E^n = E_n,\ k=0$$

$k = 2 \qquad (E_0 = 1,\ E_1 = 0)$

$0 = E_2 + 2E + 1 + E^2 - 2E + 1 = 2(E_2 + 1)$

$E_2 = -1$

$E_4 = 5$

$E_6 = -61$

$E_8 = 1385$

2-12. $$k\pi \sec k\pi = 4k\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2-4k^2}$$

$$= 4k\sum_{0} \frac{(-)^n}{(2n+1)}\frac{1}{1-\left(\frac{2}{2n+1}k\right)^2}$$

$$= 4k\sum_{n=0} \frac{(-1)^n}{(2n+1)}\sum_{m=0}^{\infty}\left(\frac{2}{2n+1}k\right)^{2m}$$

$$= 4k\sum_{n\,0}\sum_{m\,0} \frac{(-1)^n}{(2n+1)}\left(\frac{2k}{2n+1}\right)^{2m}$$

$$\Rightarrow \sec k\pi = \frac{4}{\pi}\sum_{n=0}\sum_{m=0}\frac{(-1)^n}{(2n+1)}\left(\frac{2k}{2n+1}\right)^{2m}$$

~~CONSIDER $r = m+n$ TH TERM (ie k^r)~~

~~COEFFICIENT IS:~~

$$\cancel{\frac{4}{\pi}\sum_{n,m=0}\frac{(-)^n}{2n+1}\left(\frac{2}{2n+1}\right)^{2m}\delta_{r,2m}}$$

CHANGE ORDER OF SUMMATION:

$$\sec k\pi = \frac{4}{\pi}\sum_{m=0}^{\infty}\underbrace{\left[\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)}\left(\frac{2}{2n+1}\right)^{2m}\right]}_{4^m Q(m)}k^{2m}$$

$$Q(m) = \sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)^{2m+1}}$$

$$= 1 - \frac{1}{3^{2m+1}} + \frac{1}{5^{2m+1}} - \dots$$

$$= \xi(2m+1)$$

$$\Rightarrow \sec k\pi = \sum_{0}\left[\frac{4^{m+1}}{\pi}\xi(2m+1)\right]k^{2m}$$

$$\Rightarrow \sec k\pi = \sum_{m=0}^{\infty}\left[\frac{4^{m+1}}{\pi}\ \xi(2m+1)\right] K^{2m}$$

BUT ALSO:

$$\cancel{\sec \pi = \sum_{m=0}^{\infty}\left[(-)^{m}\pi^{2m}\frac{E_{2m}}{(2m)!}\right]K^{2m}}$$

$$\sec z = \sum_{0}^{\infty}(-)^{m}\ \frac{E_{2m}}{2m!}\ z^{2m}$$

$$\Rightarrow \sec k\pi = \sum_{0}\left[(-1)^{m}\ \pi^{2m}\ \frac{E_{2m}}{(2m)!}\right]K^{2m}$$

EQUATING GIVES

$$\xi(2m+1) = \frac{(-)^{m}\ \pi^{2m}\ \frac{E_{2m}}{(2m)!}}{4^{m+1}/\pi}$$

$$= \frac{\pi}{4}(-)^{m}\left(\frac{\pi}{2}\right)^{2m}\frac{E_{2m}}{(2m)!}$$

THUS

(1) $S_1 = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \xi(3)$

$m=1$

$$= -\frac{\pi}{4}\left(\frac{\pi}{2}\right)\frac{E_2}{(2m)!}$$

$$E_2 = -1$$

$$\Rightarrow S_1 = \frac{\pi^2}{16}$$

(2) $S_2 = \xi(2n+1) = \frac{\pi}{4}(-)^{n}\left(\frac{\pi}{2}\right)\frac{E_{2n}}{(2n)!}$

9-2-75 (TUES)

DR. JOHN D. REICHERT

9-4-75 (THURS)

9-9-75 (TUES)

METHODS OF D.E. SOLUTION

(1) SEPERABLE (ONLY FOR FIRST ORDER EQUATION)

$A(x)dx + B(Y)dY = 0$ FROM $Y'B(Y) + A(x) = 0$

$\int A(x)dx + \int B(Y)dY = \int 0$

EX: $YY' + X = 0$

$YdY + XdX = 0$

$\int YdY + \int XdX = 0$

$\frac{X^2+Y^2}{2} = C' \Rightarrow X^2 + Y^2 = C$

PUTTING IN LIMITS

$\int_{X_0}^{X} A(x)dx + \int_{Y_0}^{Y} B(x)dX = C \ni Y(X_0) = Y_0$

(2) EXACT

$A(X,Y)\,dX + B(X,Y)\,dY = 0$

$dU(X,Y) = \frac{\delta U}{\delta X}dX + \frac{\delta U}{\delta Y}dY = A(X,Y)dX + B(X,Y)dY = 0$

IF $A = \frac{\delta U}{\delta X}$ $B = \frac{\delta U}{dY} \Rightarrow \frac{dA}{dY} = \frac{dB}{dX} = \frac{\delta^2 U}{\delta X\,\delta Y}$

SO RUN TEST: IS $\frac{\delta A}{\delta Y} = \frac{\delta B}{\delta X}$? IF SO, D.E IS CRACKED

EX: $A(X,Y) = X^2Y^3$ $B(X,Y) = X^3Y^2$

$\frac{\delta A}{\delta Y} = 3X^2Y^2$ $\frac{\delta B}{\delta X} = 3X^2Y^2$

$X^3Y^2Y' + X^2Y^3 = 0$

$X^3Y^2dY + X^2Y^3dX = 0$ EXACT (BUT ALSO SEPERABLE)

EX. $\sin Y\,dX + X\cos Y\,dY = 0$ ⇐ EXACT, SINCE $\cos Y = \cos Y$

$\frac{\delta U}{dX} = \sin Y$ | $\frac{\delta U}{\delta Y} = X\cos Y$

$U = X\sin Y + f(Y)$ | $U = X\sin Y + g(X)$

$U = U \Rightarrow f(Y) = g(X) = C$

⇒ ANSWER IS $X\sin Y = C$

EX. $dX + X\cot Y\,dY = 0$ ⇐ NOT EXACT,

WILL BE THOUGH IF MULTIPLIED

THROUGH BY $\lambda(X) = \sin X$

(3) INTEGRATING FACTOR $\left\{ \begin{array}{l} \text{GUESS} \\ \text{FIND A MACHINE} \end{array} \right.$

$$A\,dx + B\,dY = 0$$
$$\lambda A\,dx + \lambda B\,dY = 0$$
$$\frac{\delta U}{\delta x} = \lambda A \qquad \frac{\delta U}{\delta Y} = \lambda B$$

IF $\lambda = \lambda(x)$

$$\frac{\delta^2 U}{\delta x \delta Y} = \lambda \frac{\delta A}{\delta Y} \qquad \frac{\delta^2 U}{\delta x \delta Y} = \lambda' B + \lambda \frac{\delta B}{\delta x}$$
$$\Rightarrow \frac{\delta \lambda}{\delta x} B + \lambda\left(\frac{\delta B}{\delta x} - \frac{\delta A}{\delta Y}\right) = 0$$

IF IT DON'T WORK, TRY A $\lambda = \lambda(Y)$

CONSIDER PREVIOUS EXAMPLE

$$\lambda\,dx + \lambda x \cot Y\,dY = 0 \quad \Rightarrow \lambda = \lambda(Y)$$
$$\frac{\delta U}{\delta x} = \lambda \quad ; \quad \frac{\delta U}{\delta Y} = \lambda x \cot Y$$
$$\frac{\delta^2 U}{\delta x\, dY} = \frac{\delta \lambda}{\delta Y} \quad ; \quad \frac{\delta^2 U}{\delta x \delta Y} = \lambda \cot Y$$
$$\frac{\delta \log \lambda}{\delta Y} = \cot Y$$
$$\Rightarrow \log \lambda = \int \cot Y\,dY + C = \log \sin Y$$
$$\Rightarrow \lambda = \sin Y$$

(4) $\frac{dY}{dx} + f(x)Y = g(x)$ $\Leftarrow$ LINEAR FIRST ORDER EQN.

HOMOGENEOUS IF $Y=0$ IS A SOLUTION.

MAY ALWAYS GET AN INTEGRATING FACTOR.

$$Y' + fY = g$$
$$\lambda(Y' + fY) = g\lambda \quad \Rightarrow \lambda = \lambda(x)$$
$$\Rightarrow \text{WE WANT } \lambda f = \lambda' \Rightarrow \lambda = e^{\int f(x)dx}$$
$$\Rightarrow \frac{d}{dx}\left[e^{\int f(x)dx}\, Y(x)\right] = g(x)\, e^{\int^x f(x)dx}$$

BECOMES SEPERABLE

(5) CHANGE OF VARIABLE

(a) $Y' = f(ax + bY + c)$

$$V = ax + bY + C \Rightarrow \frac{dV}{dx} = a + b\frac{dY}{dx} \Rightarrow \frac{dY}{dx} = \frac{1}{b}\left[\frac{dV}{dx} - a\right]$$

GIVES $\frac{dV}{dx} = a + b f(V)$ $\Leftarrow$ SEPERABLE

(b) BERNOULLI

$$Y' + f(x)Y = g(x)Y^n \quad \Rightarrow \frac{Y'}{Y^n} + f(x)/Y^{n-1} = g(x)$$

LET $V = \frac{1}{Y^{n-1}}$ WILL MAKE IT LINEAR

9-16-75 (TUES.)

HOMEWORK ANSWERS

1-1. $X^2Y' + Y^2 = XYY'$

LET $Y = VX$ MAKES IT ISOBARIC

GIVES $\frac{V'}{V} - V' + \frac{1}{X} = 0$

$\left(\frac{1}{V} - 1\right)dV + \frac{1}{X}dX = 0$ ⇐ SEPERABLE!

1-2. $Y' = \frac{X\sqrt{1+Y^2}}{Y\sqrt{1+X^2}}$

$\frac{YY'}{\sqrt{1+Y^2}} = \frac{X}{\sqrt{1+X^2}}$

$W = Y^2,\ Z = X^2 \Rightarrow \frac{dZ}{\sqrt{1+Z}} = \frac{dW}{\sqrt{1+W}}$

1-3. $Y' = \frac{a^2}{(X+Y)^2}$

LET $W = X + Y$

$\frac{dW}{dX} = 1 + \frac{a^2}{W^2}$ ⇐ SEPERABLE!

DEVIATION (1): $Y = \sinh X = \frac{e^X - e^{-X}}{2}$

LET $e^X = W \Rightarrow X = \ln W$

FROM $W^2 - 2YW - 1 = 0 \Rightarrow 2W = 2Y \pm \sqrt{(2Y)^2 + 4}$

$= Y \pm \sqrt{Y^2 + 1}$

(2): $e^{iX} = \cos X + i \sin X$

$(e^X)^i$

$(e^X e^{2\pi i n})^i$

$e^{iX} e^{-2\pi n} \Rightarrow e^{iX}$ IS NOT A FUNCTION

UNDERSTOOD, USE $n = 0$ BRANCH

1-4. $Y' + Y\cos X = \frac{1}{2}\sin 2X$

HOMOGENEOUS SOLUTION IS $Ce^{-\sin X}$

MAY USE INTEGRATING FACTOR

1-5. $(1 - X^2)Y' - XY = XY^2$ ⇐ SEPERABLE!

$\frac{dY}{Y(1+Y)} = \frac{X\,dX}{1 - X^2}$

1-6. $2X^3Y' = 1 + \sqrt{1 + 4X^2Y}$

LET $Y = V/X^2 \Rightarrow 2XV' - 4V = 1 + \sqrt{1+4V}$ ⇐ SEP!

$\Rightarrow \frac{dV}{(1+4V) + \sqrt{1+4V}} = \frac{dX}{2X}$

(1-7) $Y'' + Y'^2 + 1 = 0$

$p = Y'$

$p' + p^2 + 1 = 0 \Leftarrow$ SEPERABLE

(1-8) $Y'' = e^Y$

$Y'Y'' = Y'e^Y$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}Y'^2 - e^Y\right) = 0$

$\Rightarrow Y' = \sqrt{2C - 2e^Y}$

(1-9) $X(1-X)Y'' + 4Y' + 2Y = 0$

TWO METHODS:

1. LET $g(x) = X(1-X)Y'$

$g'(x) = X(1-X)Y'' + (1-X)Y' - XY'$

$\Rightarrow X(1-X)Y'' = g' - (1-2X)Y'$

$\therefore g' - (1-2X)Y' + 4Y' + 2Y = 0$

$g' + (2X+3)Y' + 2Y = 0$

$f(x) = (2X+3)Y$

$\Rightarrow f' = (2X+3)Y' + 2Y$

GIVES $g' = f'$

$g + f = C$

OR $X(1-X)Y' + (2X+3)Y = C$

GET AN INTEGRATING FACTOR

OR HOMO + PARTICULAR

2. TRY TO GET A SPECIAL SOLUTION OF THE FORM $Y = AX^2 + BX + C$

FIND THAT $p(x) = X^2 - 5X + 10$ WORKS.

THEN LET $Y = Vp$ TO GET

$V'' + \left[\frac{2X-5}{X^2-5X+10} + \frac{4}{X(1-X)}\right]V' = 0$

LET $V' = \gamma$

AND GET A SEPERABLE EQUATION FOR γ

9-18-75 (THURS.)

(1-10) $(1-x)Y^2 dx - x^3 dY = 0 \Leftarrow$ SEPERABLE

(1-11) $xY' + Y + x^4 Y^4 e^x = 0 \Leftarrow$ BEROULLI EQN.

$$Y' + \frac{1}{x} Y = \left[-x^3 e^x\right] Y^4$$

$$V = Y^{1-4} = Y^{-3} \Rightarrow Y = V^{-3}$$

$$V' - \frac{3}{x} V = -x^3 e^x$$

$$V = V_H + V_p$$

$$V_H' - \frac{3}{x} V_H = 0$$

$$\xrightarrow{V} \boxed{D - 3/x} \xrightarrow{-x^3 e^x}$$

TRY $V_p = A x^3 e^x$ $\quad A = 3$ WORKS

$$V_p = 3x^3 e^x$$

(1-12) $(1+x^2)Y' + Y = \tan^{-1} x \Leftarrow$ LINEAR

$$Y = Y_H + Y_p$$

$(1+x^2)Y_H' + Y_H = 0 \Leftarrow$ SEPERABLE

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$Y_p = \tan^{-1} x - 1$$

(1-13) $X^2 Y'^2 - 2(xY - 4)Y' + Y^2 = 0$

$$(XY')^2 - 2XYY' + 8Y' + Y^2 = 0$$

$$[XY' - Y]^2 + 8Y' = 0$$

$(XY' - Y) = \pm\sqrt{-8Y'} \Leftarrow$ CLAIROUT

$$Y' + XY'' - Y' = \pm \frac{d}{dx}\sqrt{-8Y'} = XY''$$

$$Y''\left[2(xY' - Y)X + 8\right] = 0$$

$Y'' \Rightarrow Y = AX + B$ $\quad$ BUT B & A GOTTA BE RELATED

$Y = \frac{B^2}{8} X + B \Leftarrow$ FROM PLUGGING IN

$$2X^2 Y' = 2XY - 8 \Rightarrow XY' = Y - \frac{4}{X}$$

GIVES $Y_p = -\frac{2}{X}$

$[D^2 - 2xD + (x^2 - 1)]\phi = g$

$\phi'' - 2x\phi' + (x^2-1)\phi = g$

$(D-x)(D-x)\phi = g$ NOTE $Dx \neq xD$

$(D-x)^2\phi = g$

$\phi = \frac{g}{(D-x)^2}$ OR LET $\psi = (D-x)\phi$

$\Rightarrow (D-x)\psi = g$

9-23-75 (TUES)

$y'' + f(x)y' + g(x)y = 0$

$y = vp$

$p = e^{-\frac{1}{2}\int f(x)\,dx}$

WILL GET YOU TO $v'' + \left[g - \frac{f^2}{4}\right]v = 0$

APPROXIMATE SOLUTIONS:

$(x-1)(x+1)(x+2) = x^3 + 2x^2 - x - 1 = 0$

1. $x = -1 + 2x^2 + x^3 \Leftarrow$ ITERATION

$x = .01$, -1.01, ...

$x = -1.01$ ETC. CONVERGES TO -1

(WILL PROBABLY GIVE YOU ONLY ONE RT.

2. $x^2 = \frac{x+1-x^3}{2} \Rightarrow x = \pm\sqrt{\frac{1+x-x^3}{2}} \Leftarrow$ ITERATION

WILL STILL HAVE SAME PROBLEMS.

3. $x = \sqrt[3]{1+x-2x^2} \Leftarrow$ ITERATION

MAY USE ON $\log x + 3x^2 + e^x = 0$ AND OTHER UGLIES

DIFFERENCE EQUATION

$S_1 : \sqrt{1}$ $\qquad T_1 = 1$

$S_2 : \sqrt{1+\sqrt{1}}$ $\qquad T_2 = 1 + \frac{1}{1}$

$S_2 : \sqrt{1+\sqrt{1+\sqrt{1}}}$ $\qquad T_3 = 1 + \frac{1}{1+1}$

$S_3 : \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}$ $\qquad T_4 = 1 + \frac{1}{1+\frac{1}{1+1}}$

$S_1 = 1 \qquad S_{n+1} = \sqrt{1+S_n}$ $\qquad T_{n+1} = 1 + \frac{1}{T_n}$

$\therefore S_\infty = \sqrt{1+S_\infty}$ $\qquad T_0 = 1$

$\Rightarrow S_\infty = \frac{1+\sqrt{5}}{2}$

$R = \frac{1+\sqrt{5}}{2} = S_\infty = t_\infty$

GENERALIZE

$$S^2 - S - X = 0$$

$$f_1 = \sqrt{X}$$

$$f_2 = \sqrt{X + \sqrt{X}}$$

$$f_3 = \sqrt{X + \sqrt{X + \sqrt{X}}}$$

$$f_\infty^2 - f - X \Rightarrow f = \frac{1 + \sqrt{1 + 4X}}{2}$$

WKB METHOD

$$Y'' + f(x)\,Y = 0$$

WORKS FOR $\frac{1}{2}|f'| \ll |f|^{3/2}$

TRY $Y'' + X^2 Y = 0$ NOTE: $|X| \ll X^3$ FOR GOOD

$\frac{1}{\sqrt{X}} e^{\pm i X^2}$ ⇐ FROM EQ'N 1-92.

TRY $Y'' + X^4 Y = 0$

$\frac{1}{X} e^{\pm i X^3/3}$ GOOD FOR $2|X^3| \ll |X|^6$

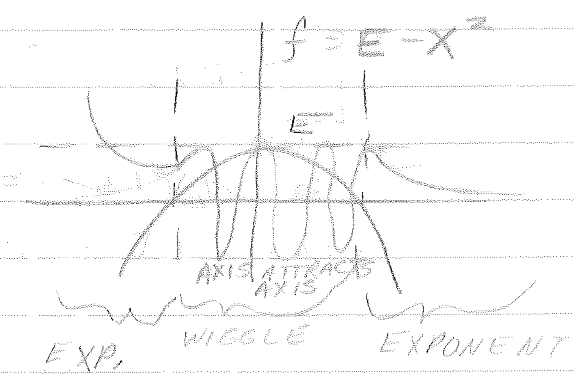
$2 \ll X^3$

TRY:

$$Y'' + (E - X^2)\,Y = 0$$

$E - X^2 = B^2 > 0 \Rightarrow Y \approx \sin B$

$E - X^2 = -B^2 < 0 \Rightarrow Y \approx e^{\pm B}$



9-25-75 (THURS)

FIELD THEORY

IF $\vec{E} = \nabla\phi$, THEN $\nabla \times E = 0$ OR $\oint E \cdot dl = 0$

EX: $E = \hat{e}_x + z e^y$

$$\nabla \times E = \begin{bmatrix} \hat{e}_x, & \hat{e}_y & \hat{e}^z \\ d_x & d_y & d_z \\ 1 & z & 0 \end{bmatrix} = -\hat{e}_x \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$F = z(x\hat{e}_x + y e^y) \Rightarrow \nabla \times F = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

IF $\nabla\phi = P\vec{E}$

$$\Rightarrow \nabla \times \nabla\phi = 0 = \nabla \times (P\vec{E})$$

$$\nabla \times (P\vec{E})$$
$$= \overset{\Delta\Delta\Delta}{Q}_{\alpha\beta\gamma}\, d_\beta (PE_\gamma) = V_\alpha$$

$$\overset{\Delta\Delta\Delta}{Q}_{\alpha\beta\gamma}\left[P_\beta E_\gamma + P\partial\beta E_\gamma\right] = \overset{\Delta\Delta\Delta}{Q}\left[P_\beta E_\gamma\right] + P\overset{\Delta\Delta\Delta}{Q}\, d_\beta E^\gamma$$

$$\left[(A \times B)_\alpha = \overset{\Delta\Delta\Delta}{Q}_{\alpha\beta\gamma} A_\beta B_\gamma \quad ; \quad (\nabla \times B)_\alpha = \overset{\Delta\Delta\Delta}{Q}_{\alpha\beta\alpha}\, d_\beta B_\alpha\right]$$

$$\overset{\Delta\Delta\Delta}{Q}\, d_\beta(PE_\gamma) = \nabla P \times E + P\nabla \times E$$

TRY $\left[\nabla \times \phi\nabla\psi\right] = \hat{Q}_{\alpha\beta\gamma}\, \partial\beta(\phi\psi) \ni \psi_\gamma = \frac{\delta\psi}{\delta X_\gamma}$

$$= \hat{Q}\left[\phi_\beta \psi_\gamma + \phi\, d_\beta \psi_\gamma\right]$$
$$= \underbrace{\hat{Q}_{\alpha\beta\gamma}\phi_\beta\psi_\gamma}_{} + \phi\hat{Q}_{\alpha\beta\gamma} d\beta\psi_\gamma$$

$$(\nabla\phi) \times (\nabla\psi) + \phi\, \cancel{\nabla \times (\nabla\psi)}^{\,0}$$

$$\Rightarrow \nabla \times \phi\nabla\psi = (\nabla\phi) \times (\nabla\psi)$$

TRY: $[\nabla \times (\vec{E} \times \vec{F})]_\alpha = Q_{\alpha} \partial_\beta (E \times F)_\gamma$

$$= Q_{\alpha\beta\gamma}\, \partial_\beta\, Q_{\gamma\mu\nu}\, E_\mu F_\nu$$
$$= Q_{\alpha\beta\gamma}\, Q_{\gamma\mu\nu}\, \partial_\beta E_\mu F_\nu$$

NOW $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$

$$\begin{vmatrix} \delta_{\alpha\mu} & \delta_{\alpha\nu} \\ \delta_{\beta\mu} & \delta_{\beta\nu} \end{vmatrix} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$$

$$\nabla \times (E \times F)_\alpha = \sum_{\beta\mu\nu} [\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}]\, \partial_\beta E_\mu F_\nu$$

$$= \sum_{\beta\mu\nu} \delta_{\alpha\mu}\delta_{\beta\nu}\, \partial_\beta E_\mu F_\nu - \sum_{\beta\mu\nu} \delta_{\alpha\nu}\delta_{\beta\mu}\, \partial_\beta E_\mu F_\nu$$

$$= \sum_\beta \partial_\beta E_\alpha F_\beta - \sum_\beta \partial_\beta E_\beta F_\alpha$$
$$= F_\beta(\partial_\beta E_\alpha) + E_\alpha \partial_\beta F_\beta - F_\alpha(\partial_\beta E_\beta) - E_\beta \partial_\beta F_\alpha$$

$\sum \partial_\beta E_\beta = (\nabla \cdot E)$

$$\Rightarrow [\nabla \times (E \times F)]_\alpha = E_\alpha(\nabla \cdot F) - F_\alpha(\nabla \cdot E)$$
$$+ F_\beta\, \partial_\beta E_\alpha - E_\beta\, \partial_\beta F_\alpha$$
$$= E_\alpha(\nabla \cdot F) - F_\alpha(\nabla \cdot E) + (F \cdot \nabla)E_\alpha - (E \cdot \nabla)F_\alpha$$

THUS

$$\vec{\nabla} \times (\vec{E} \times \vec{F}) = \vec{E}(\nabla \cdot \vec{F}) - \vec{F}(\vec{\nabla} \cdot \vec{E}) + (\vec{F} \cdot \vec{\nabla})\vec{E} - (\vec{E} \cdot \nabla)\vec{F}$$

CONSIDER $Q_{\alpha\beta\gamma}\, Q_{\gamma\beta\nu}\, E_\alpha F_\nu$

$$Q_{\alpha\beta\gamma}\, Q_{\gamma\beta\nu} = \begin{bmatrix} \delta_{\alpha\beta} & \delta_{\alpha\nu} \\ \delta_{\beta\beta} & \delta_{\beta\nu} \end{bmatrix} = \delta_{\alpha\beta}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\beta}$$

$\delta_{\beta\beta} = \sum_1^3 1 = 3$ $\qquad = \delta_{\alpha\nu} - 3\delta_{\alpha\nu}$

$$A \times B = -B \times A \Rightarrow Q_{\alpha\beta\gamma} A_\beta B_\gamma = -Q_{\alpha\beta\gamma} B_\beta A_\gamma$$
$$= -Q_{\alpha\mu\nu} B_\mu A_\nu$$
$$= -Q_{\alpha\gamma\beta} B_\gamma A_\beta$$
$$= -Q_{\alpha\gamma\beta} A_\beta B_\gamma$$

$$[Q_{\alpha\beta\gamma} + Q_{\alpha\gamma\beta}] A_\beta B_\gamma = 0$$
$$\Rightarrow Q_{\alpha\beta\gamma} = Q_{\alpha\gamma\beta}$$

$$Q_{\alpha\beta\gamma} = \epsilon^{123}_{\alpha\beta\gamma} Q_{123} \ni \epsilon^{123}_{\alpha\beta\gamma} = \begin{cases} 1 & \text{IF } \alpha\beta\gamma \text{ IS A CYCLIC PER. OF } 123 \\ -1 & \text{" " " " ODD " " "} \\ 0 & \text{OTHERWISE} \end{cases}$$

$(A \times B)_\alpha = \epsilon_{\alpha\beta\gamma} A_\beta B_\gamma$

$\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu} = \begin{bmatrix} \delta_{\alpha\mu} & \delta_{\alpha\nu} \\ \delta_{\beta\mu} & \delta_{\beta\nu} \end{bmatrix}$

RECALL: $E \times \nabla P = P\,\vec{\nabla} \times E$ IF $\nabla\phi = P\vec{E}$

$E \times \nabla\Gamma = \vec{\nabla} \times E \Rightarrow \Gamma = \log P$

$E \cdot (\nabla \times E) = 0$ IS NECESSARY

SOLVE: $\vec{F} = z(x\hat{e}_x + y\hat{e}_y)$

FIND $\phi \ni \nabla\phi = P\vec{F}$

9-30-75 (TUES)

1-14. $\underbrace{YY'' - Y'Y'}{} - 6XY^2 = 0$

$Y^2 \frac{d}{dx}\left(\frac{Y'}{Y}\right) - 6XY^2 = 0$

$\frac{d}{dx}\left[\frac{Y'}{Y} - 3X^2\right] = 0$

$\Rightarrow \ln Y = X^3 + CX + C_2$

$Y = De^{X^3 + CX}$

1-15. $\underbrace{X^4Y'' + X^4Y'Y'}{} + 3X^3YY' - 1 = 0$

$\frac{1}{2}X^4 \frac{d}{dx}(2YY') + \frac{3}{2}X^3(2YY') - 1 = 0$

$Z = 2YY'$

$\Rightarrow \frac{X^4}{2}Z' + \frac{3}{2}X^3 Z = 1$

$\Rightarrow Z' + \frac{3}{X}Z = 2/X^4$

$\frac{1}{X^3}\frac{d}{dX}(X^3 Z) = 2/X^4$

$d(X^3 Z) = \frac{2}{X}dx$

$\Rightarrow X^3 Z = 2\ln CX$

1-16. $X^2Y'' - 2Y = X$

$Y_p = -\frac{1}{2}X$

METHODS FOR HOMO

① $Y = X^m$

$\Rightarrow m = 2, -1$

$Y_H = AX^2 + \frac{B}{X}$

② USE $Y_p = -\frac{1}{2}X$ & USE V.P. TRICK

$P = X^2$

GIVES $V'' + \frac{4}{X}V' = 0$

$q = V'$ GIVES A SEPERABLE EQN.

1-17. $Y''' - 2Y'' - Y' + 2Y = \sin X$

$Y_p = A\sin X + B\cos X$

$A = \frac{1}{5}$, $B = \frac{1}{10}$

HOMO SOLUTION

$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\lambda = 1 \Rightarrow e^x$

$(\lambda - 1)(\lambda^2 - \lambda - 2) = 0 = (\lambda - 1)(\lambda + 1)(\lambda - 2)$

$\Rightarrow Y_H = Ae^x + Be^{2x} + Ce^{-x} + Y_p$

1-18. $Y^{IV} + 2Y'' + Y = \cos X$

HOMO SOLUTION: $Y_H = e^{\lambda x}$

$\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = 0$

$\Rightarrow \lambda = \pm i, \pm i$

$\Rightarrow Y_H = Ae^{ix} + Be^{-ix} + Cxe^{ix} + Dxe^{-ix}$

FORCING TERM COMBINATION OF HOMO SOLUTION

TRY: $Ax^2\cos x$

GIVES $A = -\frac{1}{8}$

1.19. $Y'' + 3Y' + 2Y = e^{e^x}$

HOMO SOLN: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -2, -1$

$$Y = \frac{e^{e^x}}{(D+1)(D+2)}$$
$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x}$$
$$= \underbrace{\frac{1}{D+1} e^{e^x}}_{\phi} - \underbrace{\frac{1}{D+2} e^{e^x}}_{\psi}$$

$(D+1)\phi = e^{e^x}$ $\qquad (D+2)\psi = e^{e^x}$

NO WORK
$$\left\{\begin{array}{ll} \phi = \alpha B & \\ (D+1)\phi = \alpha' B + \alpha B' + \alpha B & \text{NO WORK} \\ z = e^x \Rightarrow \frac{1}{D_z + 1} e^z = \phi & \\ \frac{1}{zD+1} e^z = \phi & \text{NO WORK TOO} \end{array}\right.$$

$e^x (D+1)\phi = e^x e^{e^x}$

$D(\phi e^x) = e^x e^{e^x}$

SIMPLER.

$$? \longrightarrow \boxed{} \longrightarrow e^{e^x}$$

TRY $e^{e^x} \Leftarrow$ NO WRK

$x^\alpha e^{e^x} \Leftarrow$ NO WORK

$e^{\alpha x} e^{e^x} \Leftarrow$ WORKS FOR $\alpha = -2$

1-20.

$a^2 y'' y'' = (1 + y' y')^3$

$p = y'$

$\Rightarrow a^2 p' p' = (1 + p^2)^3$

$\frac{\pm a\,dp}{(1+p^2)^{3/2}} = dx$ ETC

1-21.

$R\frac{dq}{dt} + \frac{1}{C}q = V_0\left(\frac{t}{\tau}\right)^2 e^{-t/\tau}$

$\frac{dq}{dx} + \alpha q = \beta x^2 e^{-x}$, $x = \frac{t}{\tau} \Rightarrow$ DIMENSIONLESS

1-22. EZ

1-23. $A(x)\,y'' + A'(x)\,y'(x) + \frac{y(x)}{A(x)} = 0$

$\frac{d}{dx}[Ay'] + y/A = 0$

$\frac{d}{dx}[Ay']^2 + 2yy' = 0$

$(Ay')^2 + y^2 = C$

GIVES $y' = \pm\sqrt{C - y^2}/A(x)$

1-24. $xy'' + 2y' + n^2 xy = \sin\omega x$

$y = vp$

$pv'' + 2\left[p' + \frac{p}{x}\right]v' + \left[p'' + \frac{2}{x}p' + n^2 p\right]v = \frac{\sin\omega x}{x}$

$p = \frac{1}{x}$

GIVES $v'' + n^2 v = \sin\omega x$

$y_H = A\sin nx + B\cos nx$

1-25. $(1-x)y'' + xy' - y = (1-x^2) \Leftarrow$ LINEAR

TRY A POLINOMIAL:

$y_p = Ax^2 + Bx + C$

GIVES $y_p = x^2 + 1$

FOR HOMO, USE VP TRICK

10-2-75 (THURS)

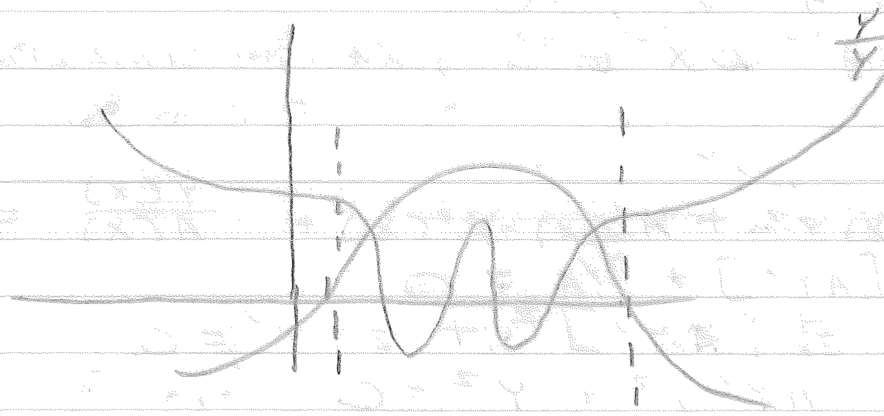
VP TRICK

CAN BRING ANY LINEAR SECOND ORDER D.E. TO: (IN HOMO. CASE:)

$$y'' + f(x)\, y = 0$$

$$Y = p y \quad : \quad p = e^{-\frac{1}{2}\int f(x)\,dx}$$

$$\frac{y''}{y} = -f(x)$$



$f(x) > 0 \Rightarrow$ Y BENDS TOWARD AXIS

$f(x) < 0 \Rightarrow$ Y BENDS AWAY FROM AXIS

HELMHOLTZ'S EQ'N

$$\nabla^2 \psi + k^2 \psi = 0 \qquad \psi = X(x)\, Y(y)\, Z(z)$$

GIVES

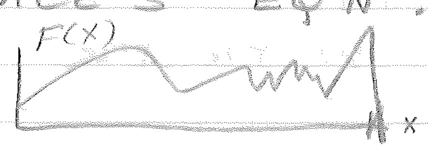
$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{= -k^2 + \alpha^2 + \beta^2} = -k^2$$

$$X'' + \alpha^2 X = 0 \qquad \sim \sin \alpha x + \cos \alpha x$$

$$Y'' + \beta^2 Y = 0$$

$$Z'' + (k^2 - \alpha^2 - \beta^2) Z = 0$$

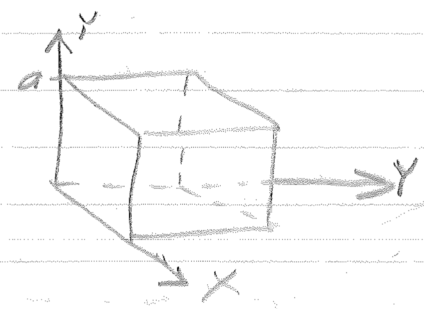
LAPLACE'S EQ'N : $k = 0$



$\Leftarrow$ FREQ. $\sim 2\pi n x / L$

MAY GENERATE ANY FUNCTIONS

CONSIDER BOUNDRY VALUE PROBLEM



$\nabla^2 \psi + k^2 \psi = 0$

B.C. $\psi = 0$ @ $Z=0$ $\forall$ X, Y ON THE FACE

ALSO ON THE SIDE FACES

$\psi = AX^2Y^2$ ON THE TOP FACE @ $Z = a$

WANT WIGGLERS IN X & Y.

DON'T NEED THEM IN Z.

SOLUTION OF FORM $X_n Y_m Z_{nm}$

PROPERTIES THAT I WANT FOR THIS PROBL.

① $X_n(0) = X_n(a) = 0 \quad \forall\, n$

② $Y_m(0) = Y_m(a) = 0 \quad \forall\, m$

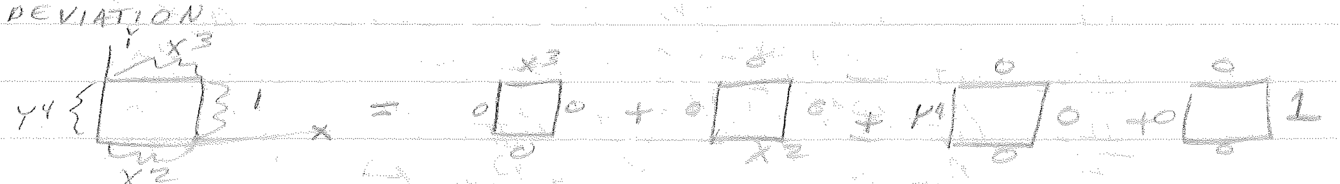
$X_n(X) = \sin \pi n\, X/a$

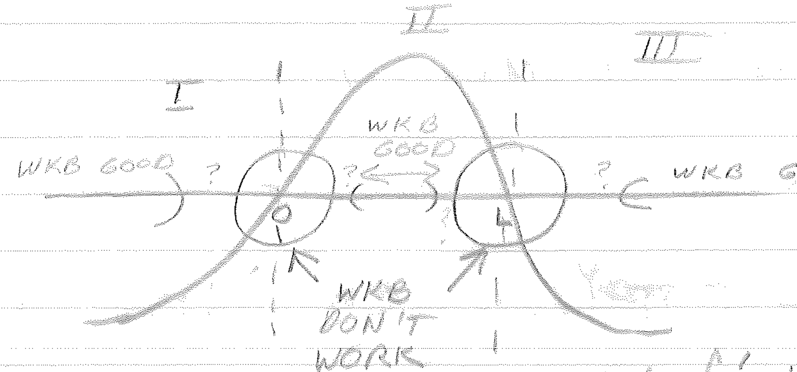
$Y_m(X) = \sin \pi m\, Y/a$

③ $Z_{nm}(0) = 0$

$$\psi(X,Y,Z) = \sum_{n,m} A_{nm} X_n(X)\, Y_m(Y)\, Z_{nm}(Z)$$

DEVIATION





$Y'' + f(x)Y = 0$

MAY USE WKB IF $\frac{1}{2}\left|\frac{f'}{\sqrt{f}}\right| \ll |f$) (EQ. 1-89)

HOPELESS FOR $f = 0$

1, $f > 0$, (I & III

$$Y \simeq \frac{1}{\sqrt[4]{f(x)}}\left[A\, e^{i\int \sqrt{f}\,dx} + B\, e^{-i\int^x \sqrt{f}\,dx}\right]$$

$$= \frac{1}{\sqrt[4]{f}}\left[A \cos \int \sqrt{f}\,dx + B \sin \int \sqrt{f(x)}\right]$$

2. $f < 0$ (II)

$$Y \approx \frac{1}{\sqrt[4]{-f}}\left\{\tilde{A}\, e^{\int^x \sqrt{-f}\,dx} + \tilde{B}\, e^{-\int^x \sqrt{-f}\,dx}\right]$$

$$= \frac{1}{\sqrt[4]{-f}}\left[\tilde{C} \cosh\left[\int^x \sqrt{-Y}\,dx\right]\right.$$

$$\left. + \tilde{D} \sinh\left(\int \sqrt{-Y}\,dx\right)\right]$$

WILL HAVE 6 ARBITRARY CONSTANTS. TO DARN MANY. REQUIRE Y AND Y' @ $X = \infty$ (Y' GOES TO 0 SINCE IT BENDS AWAY FROM AXIS,

I. $A_I\, e^{\int \sqrt{-f}\,dx} + B_I\, e^{-\int^x \sqrt{-f}\,dx}$

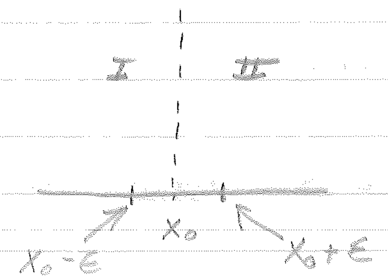
II. $A_{II}\, e^{i\int \sqrt{f}\,dx} + B_{II}\, e^{-i\int \sqrt{f}\,dx}$

III. $A_{III}\, e^{\int + \sqrt{-f}\,dx} + B_{III}\, e^{-\int \sqrt{-f}\,dx}$

TO MAKE Y GO TO ZERO,

LET $A_{III} = B_I = 0$

$Y'' + fY = 0$



$$\Leftarrow \int_{x_0-\epsilon}^{x+\epsilon} Y''\,dx + \int_{x_0-\epsilon}^{x_0+\epsilon} fY\,dx = 0$$

$$Y'\Big|_{x_0-\epsilon}^{x+\epsilon} + 0 = 0$$

$$\Rightarrow Y'_{x_0+\epsilon} = Y'_{x_0-\epsilon}$$

$\therefore$ DERIVATIVE (Y) IS CONTINUOUS

AND Y IS CONTINUOUS

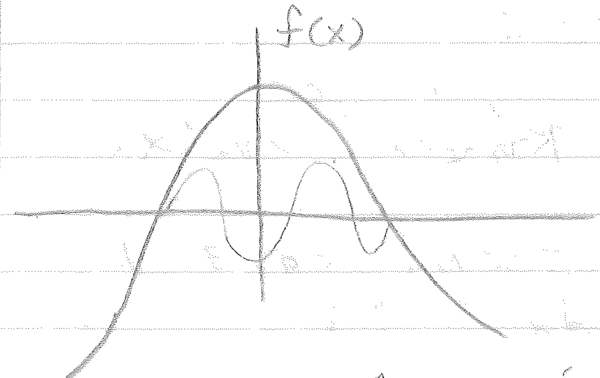
$Y \Rightarrow @x=0,\ A_I = A_{II} + B_{II}$ WLOG, $A_I = 1$

$$\Rightarrow A_{II} + B_{II} = 1$$

$Y' \Rightarrow \sqrt{-f(0)} = i\sqrt{f(0)}\,A_{II} - i\sqrt{f(0)}\,B_{II}$ } BUT 'DIS DON WORK

$$f(0) = 0$$

10-6-75 (TUES)



$Y'' + f(x)Y = 0$

$\sqrt{f(x)} \sim$ FREQUENCY (KIND OF)

$\sqrt{f(x)} = 2\pi/\lambda$

$\lambda = 2\pi/\sqrt{f}$

$$Y_{WKB} = \frac{A}{\sqrt[4]{|f(x)|}}\, e^{i\int^x \sqrt{f(x)}\,dx}$$

$$+ \frac{B}{\sqrt[4]{|f(x)|}}\, e^{-i\int^x \sqrt{f(x)}\,dx}$$

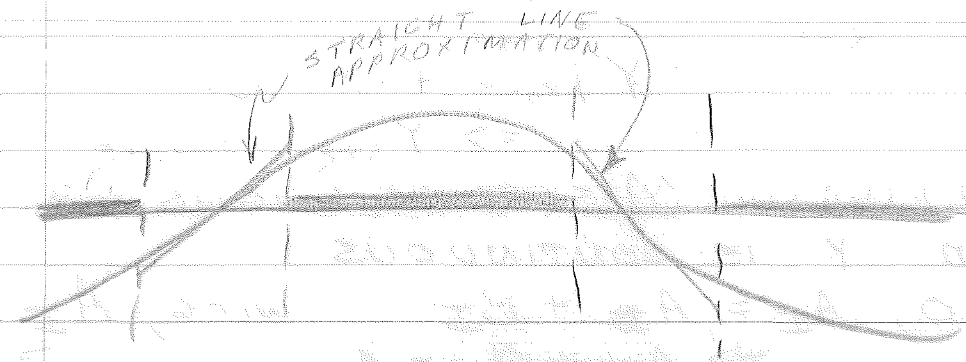
$$= \frac{\tilde{A}}{\sqrt[4]{f(x)}} \cos\left[\int^x \sqrt{f}\,dx\right]$$

$$+ \frac{B}{\sqrt[4]{f(x)}} \sin\left[\int^x \sqrt{f}\,dx\right]$$

$$Y_{WKB} = \frac{\tilde{A}}{\sqrt[4]{|f|}} \cos\left[\int^x \sqrt{f}\,dx' + \delta\right]$$

WKB = SADDLE POINT METHOD

<u>MATCHING AT BOUNDRIES FOR WKB</u>



WKB GOOD FOR 1. SHALLOW SLOPES

2. $|f| \gg 0$

USING STRAIGHT LINE FIT:

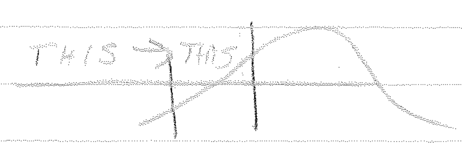
$$Y'' + (ax + b)Y = 0$$

GIVES BESSEL FUNCTIONS OF ORDER $\frac{1}{3}$: $K_{1/3}(\alpha x)$, $J_{1/3}(\alpha x)$

MATHEW'S CONNECTION (Eq. 34)

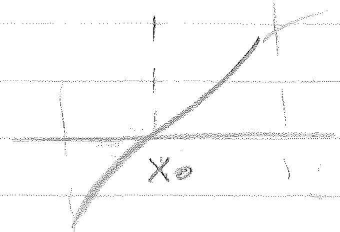
$$\frac{1}{\sqrt[4]{-f(x)}} e^{-\int_x^{x_0} \sqrt{-f(x)}\,dx}$$

$$\rightarrow \frac{2}{\sqrt[4]{f(x)}} \cos \int_{x_0}^{x} \sqrt{f(x)}\,dx - \frac{\pi}{4}$$

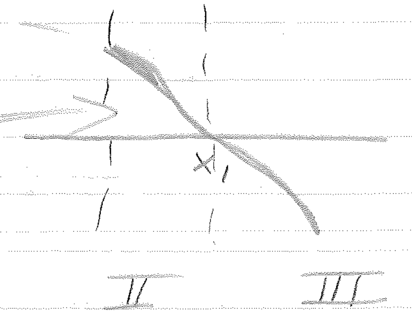


ALSO SEE Eq. 122

IN MATHEWS, WE GOT



WHAT ABOUT $\Longrightarrow$

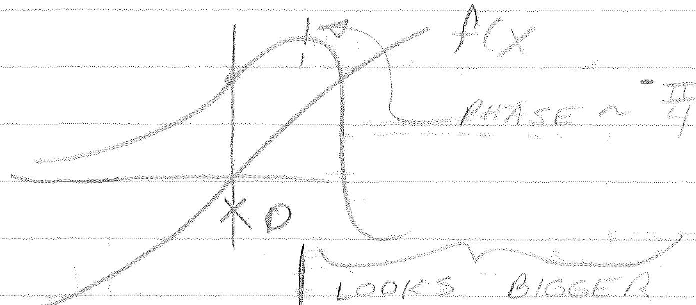


?

III $\quad \dfrac{1}{\sqrt[4]{-f}}\, e^{-\int_{x_1}^{x} \sqrt{-f}\, dx}$

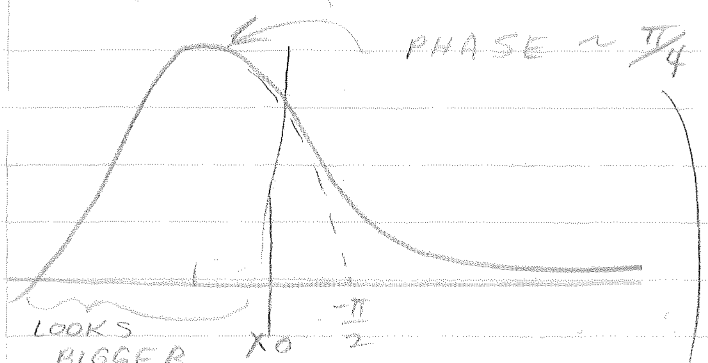
$$\longrightarrow \frac{A}{\sqrt[4]{f}} \cos\left[\int_x^{x_1} \sqrt{f}\, dx' + \delta_{sp}\right] \quad \text{II}$$

OR, $\sqrt[4]{f} \cos\left[\int_x^{x_1} \sqrt{f}\, dx + \delta\right] \longrightarrow \dfrac{(\ \ )}{\sqrt[4]{-f}}\, e^{\int_{x_1}^{x} \sqrt{-f}\, dx}$



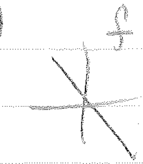
LOOKS BIGGER HERE $\Longrightarrow$ SCALING OF 2.

WHAT ABOUT II – III :



$$\frac{2}{\sqrt[4]{f}} \cos\left[\int_x^{x_1} \sqrt{f}\, dx - \frac{\pi}{4}\right] \longleftarrow \frac{1}{\sqrt[4]{f}}\, e^{-\int_{x_1}^{x} \sqrt{-f}\, dx}$$

GOING OTHER WAY (GENERAL)



$$\frac{1}{\sqrt[4]{f}} \cos \int_x^{x_1} \sqrt{f}\, dx + \delta$$

$$\longrightarrow \frac{\sin \delta + \frac{\pi}{4}}{\sqrt[4]{-f}}\, e^{\int_{x_1}^{x} \sqrt{-f}\, dx}$$

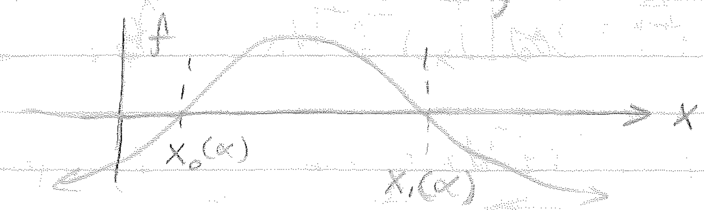
CONNECTION METHODS

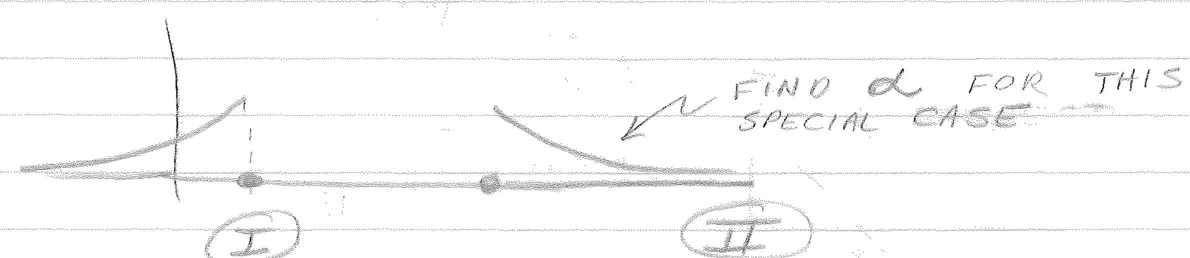
1. W_+, W_-

2. LINEAR FIT (AIRY)

DUE THURS: $f(x, \alpha)$

$$Y'' + f(x, \alpha) Y = 0$$





$$\frac{1}{\sqrt[4]{-f}}\, e^{-\int_x^{x_0} \sqrt{-f}\, dx} \longrightarrow \frac{2}{\sqrt[4]{f}} \cos\left[\int_{x_0}^{x} \sqrt{f}\, dx - \frac{\pi}{4}\right]$$

$$\frac{\sin(\delta + \frac{\pi}{4})}{\sqrt[4]{-f}}\, e^{\int_x^{x_0} \sqrt{-f}\, dx} \longleftarrow \frac{1}{\sqrt[4]{f}} \cos\left[\int_{x_0}^{x} \sqrt{f}\, dx + \delta\right]$$

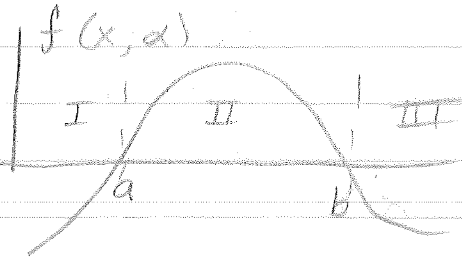
III IV

$$\frac{2}{\sqrt[4]{f}} \cos\left[\int_x^{x_1} \sqrt{f}\, dx - \frac{\pi}{4}\right] \longleftarrow \frac{1}{\sqrt[4]{-f}}\, e^{-\int_{x_1}^{x} \sqrt{-f}\, dx}$$

$$\frac{1}{\sqrt[4]{f}} \cos\left[\int_x^{x_1} \sqrt{f}\, dx + \delta\right] \longrightarrow \frac{\sin \delta + \frac{\pi}{4}}{\sqrt[4]{-f}}\, e^{\int_{x_1}^{x} \sqrt{-f}\, dx}$$

10-9-75 (THURS)

HOMEWORK



$$\int_a^b \sqrt{f}\,dx = (n+\tfrac{1}{2})\pi$$

I → II

$$\frac{1}{\sqrt[4]{-f}}\, e^{-\int_x^a \sqrt{-f}\,dx} \longrightarrow \frac{2}{\sqrt[4]{f}} \cos\left[\int_a^x \sqrt{f}\,dx - \frac{\pi}{4}\right]$$

III → II

$$\frac{1}{\sqrt[4]{-f}}\, e^{-\int_b^x \sqrt{-f}\,dx} \longleftarrow \frac{2}{\sqrt[4]{f}} \cos\left[\int_x^b \sqrt{f}\,dx - \frac{\pi}{4}\right]$$

∴ FOR Y TO BE BOUNDED

$$\cos\left[\int_a^x \sqrt{f}\,dx - \frac{\pi}{4}\right] = \cos\left[\int_x^b \sqrt{f}\,dx - \frac{\pi}{4}\right]$$

$$\cos\theta = \cos\phi \Rightarrow \phi = \pm\theta + 2\pi N$$

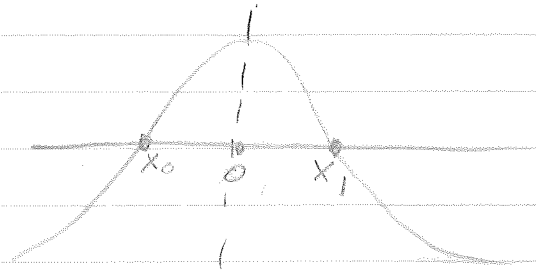
$$\int_a^x \sqrt{f}\,dx - \frac{\pi}{4} = \pm \int_x^b \sqrt{f}\,dx \mp \frac{\pi}{4} + 2\pi N$$

$$\int_a^x \mp \int_x^b = 2\pi N + \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}$$

LOWER SIGN GIVES

$$\int_a^b \sqrt{f}\,dx = 2\pi N + \frac{\pi}{2} = (2N+\tfrac{1}{2})\pi$$

(OWN WORK)



$Y'' + f(x)\,Y = 0$

$f(-x) = \pm f(x)$

THEN IF $Y(x)$ IS A SOLUTION,

SO IS $\pm Y(-x)$

$Y(x) + Y(-x) = Y_+$

$Y(x) - Y(-x) = Y_-$

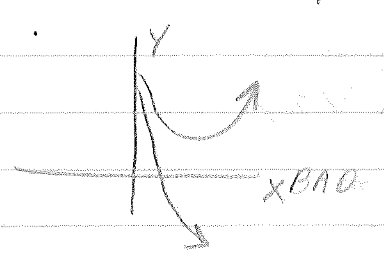
$Y(0) = 1$, $Y'(0) = 0$ $\Big\} \Leftarrow$ WILL LOOK FOR EVEN SOLUTION

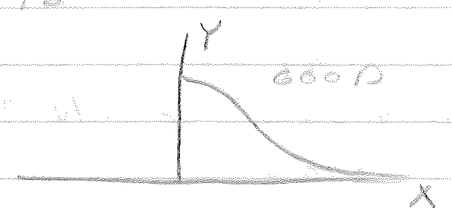
$Y(0) = 0$, $Y'(1) = 1$ $\Big\} \Leftarrow$ WILL LOOK FOR ODD MODES

INPUT α

PRINT

X	Y
0	1
.01	.98



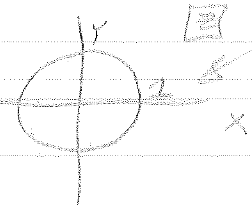


10-14-75 (TUES)

CHAPT. 2

SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = A(x)$$



CONVERGES IN HERE

BUT DON'T WORRY 'BOUT IT.

$$A(3) = \frac{1}{2} = 1 + 3 + 3^2 + 3^3 + \ldots = \frac{1}{1-3}$$

$$B(x) = 1 - x$$

$$B(3) = 1 - 3 = -2$$

$$A(x) \cdot B(x) = ?$$

$$= (1 + x + x^2 + x^3 + \ldots)(1 - x)$$

$$= \begin{array}{l} 1 + x + x^2 + x^3 + x^4 + \ldots \\ - x - x^2 - x^3 - x^4 + \ldots \end{array} = 1$$

$$e^x = \sum_0 \frac{x^n}{n!}$$

$$e^{i\theta} = \sum_0 \frac{(i\theta)^n}{n!} = \sum_{n\ \text{even}} \frac{(i\theta_n)}{n!} + \sum_{n\ \text{odd}} \frac{(i\theta_n)^n}{n!}$$

$$\sum_{n\ \text{EVEN}} f(n) = \sum_{\text{ALL}\ n} f(2n)$$

$$\sum_{n\ \text{ODD}} f(n) = \sum_{\text{ALL}} f(2n+1)$$

$$\Rightarrow e^{i\theta} = \sum_{n=0} \frac{(i\theta_n)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0} \frac{(i^2\theta^2)^n}{2n!} + \sum_{n=0}^{\infty} \frac{i\theta(i^2\theta_n^2)^n}{2n+1}$$

$$= \sum_{n=0} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0} (-)^n \frac{\theta^{2n+1}}{(2n+1)!}$$

$$= \cos\theta + i \sin\theta$$

TAYLOR EXPANSION:

$$f(z) = \sum_n \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

MACLURIN: $f(z) = \sum_n \frac{1}{n!} f^{(n)}(0)\, z^n$

CHANGES OF VARIABLES:

$$S(x) = \sum_{n=3}^{\infty} (n+7)^2/(n-1)!\; X^n$$

OBJECTIVE (BEGIN @ $n=0$)

LET $m = n - 3$

$$\Rightarrow S(x) = \sum_{m=0}^{\infty} \frac{(m+10)^2}{(m+2)!} X^{(m+3)}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{x+1}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots$$

$$-\ln(1-x) = \int \frac{dx}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots$$

CONSIDER

$$S(x) = \frac{x}{4} - \frac{x^2}{5} + \frac{x^3}{6} - \frac{x^4}{7} + \frac{x^5}{8} \ldots$$

$$x^3 \cdot S(x) = \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^3}{6} - \frac{x^4}{7} + \ldots$$

$$D[x^3 \cdot S(x)] = x^3[1 - x + x^2 - x^3] = \frac{x^3}{1+x}$$

$$\frac{d(x^3 S)}{dx} = \frac{x^3}{1+x}$$

$$x^3 S = \int_0^x \frac{\gamma^3}{1+\gamma}\,d\gamma \Rightarrow S(x) = \frac{1}{x^3}\int_0^x \frac{\gamma^3}{1+\gamma}\,d\gamma$$

CONSIDER $(1+x)^{\alpha}$

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r \ ; \ (1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

$$(-7)! = (-7)(-8)(-9) \cdots = \infty$$

$$\int_0^{\infty} x^n e^{-x}\,dx = n!$$

$$\int_0^{\infty} \sqrt{x}\, e^{-x}\,dx = \left(\tfrac{1}{2}\right)!$$

$$(1+x)^{\frac{1}{2}} = \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} x^r$$

$$\binom{\frac{1}{2}}{r} = \frac{\frac{1}{2}!}{r!\left(\frac{1}{2}-r\right)!}$$

$$\binom{\frac{1}{2}}{2} = \binom{\frac{1}{2}}{2} = \frac{\left(\frac{1}{2}\right)!}{2\left(-\frac{3}{2}\right)!} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cancel{\left(-\frac{3}{2}\right)}\cdots}{2\cancel{\left(-\frac{3}{2}\right)}}$$

$$= -\frac{1}{8}$$

9-16-74 (THURS.)

THE LOWLY FACTORIAL

NOTATIONS: $\binom{n}{r}$, $(\alpha)_n$, $\alpha!!$

• DOUBLE FACTORIAL : $\alpha!!$

$$7!! = 7\cdot 5\cdot 3\cdot 1 = \frac{7!}{6\cdot 4\cdot 2} = \frac{7!}{6!!} = \frac{7!}{2^3\, 3!}$$

$$8!! = 8\cdot 6\cdot 4\cdot 2$$

$$\Rightarrow (2n!+1)!!\,(2n)!! = (2n+1)!$$

$$2n!! = 2^n\, n!$$

$$(2n+1)!! = (2n+1)!/2^n n!$$

• BINOMIAL COEFFICIENTS: $\binom{n}{r}$

$$\binom{\alpha}{0} = \binom{\alpha}{\alpha} = 1$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$(1+x)^\alpha = \sum_{r=0} \binom{\alpha}{r} x^r$$

$$(a+b)^\alpha = \sum_{r=0} \binom{\alpha}{r} a^r b^{\alpha-r}$$

$1+x^3 = 1+3x+3x^2+x^3$

$1+x^4 = 1+4x+6x^2+4x^3+x^4$

PASCAL'S TRIANGLE

r →

↓ n

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

SAYS $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$

r	0	1	2	
$\binom{1/2}{r}$	1	$\frac{1}{2}$	$-\frac{1}{8}$	
$\binom{3/2}{r}$	1	$\frac{3}{2}$	$\frac{3}{8}$	$\ldots$

$$\binom{\alpha}{r} = \frac{\alpha!}{r!(\alpha-r)!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-r+1)(\alpha-r)!}{r!\,(\alpha-r)!}$$

$$\binom{-1}{r} = \frac{(-1)!}{r!(-1-r)!}$$

$$\binom{\alpha}{r} = \frac{\alpha}{r}\cdot\frac{\alpha-1}{r-1}\cdot\frac{\alpha-2}{r-2}\cdots\frac{\alpha-r+1}{1} \Leftarrow r \text{ FACTORS}$$

$$(\alpha)_3 = (\alpha)(\alpha+1)(\alpha+3) \Leftarrow \text{THREE FACTORS}$$

$$\Rightarrow \binom{\alpha}{r} = \frac{(\alpha-r+1)_r}{r!}$$

$$(\alpha-r+1)_r = r!\binom{\alpha}{r}$$

$$z_r = r!\binom{z+r-1}{r}$$

- $n! \equiv \underline{n|}$

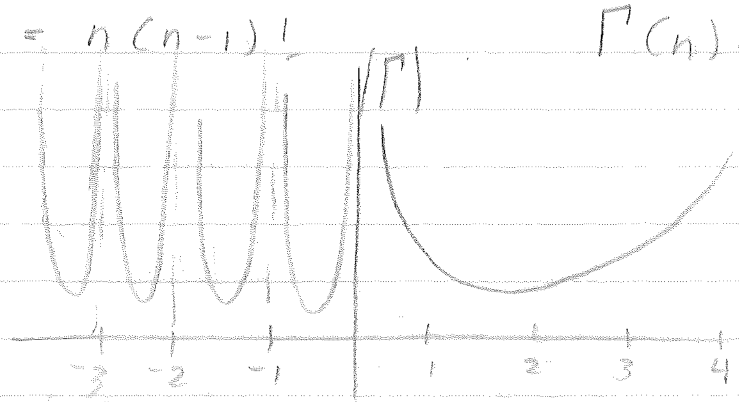
  $\equiv \Pi\, n$ = PRODUCT FUNCTION

- GAMMA FUNCTION

$$\Gamma(n+1) = n! \Rightarrow \Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx = (n-1)!$$

$$n! = \int_0^\infty x^n e^{-x}\,dx = \Pi(n)$$

$n! = n(n-1)!$ $\qquad\qquad$ $\Gamma(n) = (n-1)\,\Gamma(n-1)$



$n! = \int_0^\infty x^n e^{-x}\,dx$

$\left(\frac{1}{2}\right)! = \int_0^\infty x^{1/2} e^{-x}\,dx$

$(-1)! = \int_0^\infty \frac{1}{x} e^{-x}\,dx = \infty$

$\left(-\frac{3}{2}\right)! = \int_0^\infty \frac{1}{x^{3/2}} e^{-x}\,dx \Leftarrow$ DON'T EXIST

$\left(-\frac{1}{2}\right)! = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx = \sqrt{\pi}$

THUS:

$n! = \int_0^\infty x^\alpha e^{-x}\,dx$ FOR $\alpha > -\frac{1}{2}$

$\Gamma(z) = \int_0^\infty x^{z-1} e^{-z}\,dz$ ONLY FOR $z > 0$

RECURSION RELATION FOR ANALYTIC CONTINUATION

$$\begin{cases} \Gamma(z)\,\Gamma(1-z) = \pi / \sin \pi z \\ \alpha!\,(-\alpha)! = \pi\alpha / \sin \pi\alpha \end{cases}$$

$$\left(-\frac{3}{2}\right)! = \frac{\frac{3}{2}\pi}{\sin \frac{3\pi}{2} \left(\frac{3}{2}\right)!}$$

$\left(-\frac{1}{2}\right)! = \frac{1}{2}\left(-\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} \Rightarrow \left(\frac{1}{2}\right)! = \frac{1}{2}\left(-\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

$\left(\frac{3}{2}\right)! = \frac{3}{2}\left(\frac{1}{2}\right)! = 3\sqrt{\pi}/4$

$\Rightarrow \left(-\frac{3}{2}\right)! = -2\sqrt{\pi}$

CONSIDER FROM $\frac{1}{(1+x)^n}$

$$\binom{-n}{r} = \frac{(-n)!}{r!\,(-n-r)!} = \underbrace{\frac{-n}{r} \cdot \frac{-n-1}{r-1} \cdot \frac{-n-2}{r-2} \cdots \frac{-(n+r-1)}{1}}_{r \text{ FACTORS}}$$

$$= \frac{(-1)^r (n)_r}{r!}$$

$$= \frac{(-)^r \left[r! \binom{n+r-1}{r}\right]}{r!}$$

$$\boxed{\binom{-n}{r} = (-)^r \binom{n+r-1}{r}}$$

ALTERNATIVE

n NOT AN INTEGER, $n > 0$, $r \in$ INTEGER

$$\binom{-n}{r} = \frac{(-n)!}{r!\,(-n-r)!}$$

$$= \frac{\Gamma(1-n)}{r!\,\Gamma(1-n-r)}$$

$$= \frac{1}{r!}\,\frac{\pi}{\Gamma(n)\sin(\pi n)}\qquad\frac{\Gamma(n+r)\sin\pi(n+r)}{\pi}$$

$$= \frac{\Gamma(n+r)}{r!\,\Gamma(n)}\;\frac{\sin\pi(n+r)}{\sin\pi n}$$

$$= (-)^r\,\frac{\Gamma(n+r)}{r!\,\Gamma(n)}$$

$$= (-)^r\,\frac{(n+r-1)!}{r!\,(n-1)!}$$

$$= (-1)^r\binom{n+r-1}{r} \Leftarrow \text{SAME ANSWER AS BEFORE}$$

$$\left(-\tfrac{1}{2}\right)! = \sqrt{\pi}$$

$$\left(\tfrac{1}{2}\right)! = \tfrac{1}{2}\sqrt{\pi}$$

$$\left(n+\tfrac{1}{2}\right)! = \underbrace{\left(n+\tfrac{1}{2}\right)\left(n-1+\tfrac{1}{2}\right)\cdots}_{n \text{ FACTORS}}\left(\tfrac{1}{2}\right)!$$

$$= \frac{1}{2^n}\,(2n+1)!!\,\left(\tfrac{1}{2}\right)!$$

$$\left(n+\tfrac{1}{2}\right)! = \frac{\sqrt{\pi}}{2^{n+1}}\,(2n+1)!!$$

CONSIDER:

$$\binom{2m-1}{m}$$

$$\sum_{m=0}^{\infty} \binom{2m-1}{m} f(x)^m$$

$$= \underbrace{\binom{-1}{0}}_{\text{UGLY}} 1 + \binom{1}{1} f(x) + \ldots$$

$$\binom{2m-1}{m} = \frac{(2m-1)!}{m!\,(m-1)!}$$

WE GOTTA FIND $\lim_{m \to 0}$

$$\frac{(2m-1)!}{m!\,(m-1)!} = \frac{\pi(2m-1)}{\sin \pi(2m-1)\,(1-2m)!} \times \frac{\sin \pi(m-1)\,(1-m)!}{\pi(m-1)}$$

$$= \underbrace{\frac{1}{m!}}_{1} \; \frac{\sin \pi(m-1)}{\sin \pi(2m-1)} \; \underbrace{\frac{2m-1}{m-1}}_{1} \; \underbrace{\frac{(1-m)!}{(1-2m)!}}_{1}$$

USE LAHOPITAL'S RULE:

$$\lim_{m \to 0} \frac{\sin \pi(m-1)}{\sin \pi(2m-1)} = \frac{1}{2}$$

$$\Rightarrow \lim_{m \to 0} \binom{2m-1}{m} = \frac{1}{2} \quad ; \binom{-1}{0} \longleftarrow \frac{1}{2}$$

ANOTHER THEOREM:

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} \Gamma(z)\ \Gamma(z+\tfrac{1}{2})$$

OR $$(2w)! = 4^w/\sqrt{\pi}\ (w-\tfrac{1}{2})!\ w!$$

$$\frac{(2w)!}{w!} = \frac{4^w}{\sqrt{\pi}}(w-\tfrac{1}{2})! \Leftarrow \text{MULTIPLICATION THEOREM}$$

NOTE

$$\binom{2m-1}{2m} = \frac{1}{2m!}\ \frac{(2m)!}{m!}$$

$$= \frac{1}{2m!}\ \frac{4^m}{\sqrt{\pi}}(m-\tfrac{1}{2})!$$

$$\lim_{m\to 0} \frac{1}{2}$$

$$\begin{cases} \Gamma(nz) = c_{\text{onst}}\,\Gamma(z)\,\Gamma(z+\frac{1}{n})\Gamma(z+\frac{2}{n}) * \dots * \Gamma(z+\frac{n-}{n} \\ \Gamma(3z) = \frac{1}{\sqrt{2\pi}}\ \frac{3^{3z}}{\sqrt{3}}\ \Gamma(z)\Gamma(z+\frac{1}{3})\Gamma(z+\frac{2}{3}) \\ (3w)! = \frac{\sqrt{3}}{2\pi}\ 3^{3w}\ w!\ (w-\frac{1}{3})!\ (w-\frac{2}{3})! \end{cases}$$

CONSIDER: $$\frac{(3w-1)!}{(w-1)!} = \underbrace{\frac{w}{3w}}\ \underbrace{\frac{(3w)!}{w!}}$$

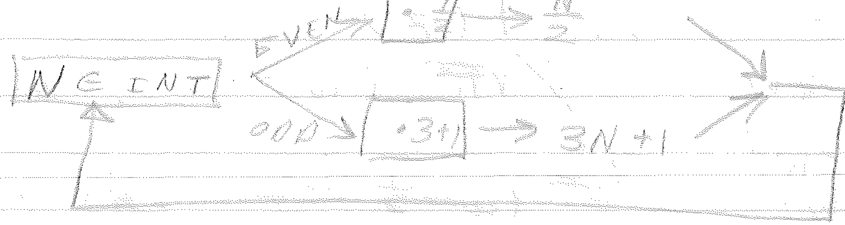
$$\lim_{w\to 0} = \left(\frac{1}{3}\right)\frac{\sqrt{3}}{2\pi}\ 3^{3w}(w-\tfrac{1}{3})(w-\tfrac{2}{3})$$

$$= \frac{1}{3}\ \frac{\sqrt{3}}{2\pi}\ (-\tfrac{1}{3})!\ (-\tfrac{2}{3})!$$

$$= \frac{3\sqrt{3}}{2\pi}\ (-\tfrac{1}{3})!\ (\tfrac{1}{3})!$$

$$= \frac{3\sqrt{3}}{2\pi}\ \frac{\pi/3}{\sin\pi/3} = \underline{1}$$

10-21-75 (TUES)



PROVE THAT $\forall N$, THE SERIES GOES TO 4, 2, 1, 4, 2, 1, ...

$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

$S = \lim_{n \to 1} \frac{x^n}{n}$

$$f'(x) = \sum x^{n-1} = \frac{1}{1-x}$$

$$f(x) - \underbrace{f(0)}_{0} = \int_0^x \frac{1}{1-x}\,dx$$

$$f(x) = -\ln(1-x)$$

$$\Rightarrow S = \lim_{x \to 1} \left[-\ln(1-x)\right]$$

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots = \frac{1}{1-\frac{1}{2}} = 2$$

REIMANN ZETA FUNCTION

WHAT IS THETA FUNCTION?

BEROLLI NUMBER

① EZ TO GENERATE

② SHOW UP REASONABLY OFTEN

$$(B+1)^n - B^n = 0 \qquad \ni B^n \equiv B_n$$

$$n=2 \Rightarrow B^2 + 2B + 1 - B^2 = 0$$

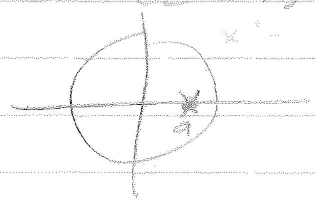
$$= 2B + 1 = 2B_1 + 1 = 0 \Rightarrow B_1 = -\frac{1}{2}$$

$$n=3 \Rightarrow (B+1)^3 = B^3 \Rightarrow B_2 = \frac{1}{6}$$

PARTIAL FRACTIONS

$f(z)$ HAS A POLE @ $z = a$

$$\mathrm{Res}\,(z=a) = \lim_{z \to a} (z-a)\, f(z)$$



$$\oint f(z)\,dz = 2\pi i\, \mathrm{Res}_a$$

$$f(z) = \frac{1}{(z-a)(z-b)}$$

POLES @ $z=a$ AND $z=b$

$$f(z) = \frac{A}{z-a} + \frac{B}{z-b}$$

$A = \mathrm{res}\ f(z)$ @ $z = a$ $= 1/a-b$

$$B = \frac{1}{b-a}$$

10-23-75 (THURS)

A HARD ONE:

$$S = \sum_{n=1}^{\infty}\left[1 - \frac{2n+n}{n}\ln\frac{n+1}{n}\right]$$

$$= \sum_{1}^{\infty}\left[x - \frac{2n+1}{2}\ln x\right]_n^{n+1}$$

$$= \sum_{1}^{\infty}\int_n^{n+1}\frac{d}{dx}\left[x - \frac{2n+1}{2}\ln x\right]$$

$$= \sum_{n=1}^{\infty}\int_n^{n+1}\frac{x-n-\frac{1}{2}}{x}\,dx$$

$$= \sum_{n=1}^{\infty}\int_n^{n+1}\frac{x-[x]-\frac{1}{2}}{x}\,dx \qquad [x] = \text{INTEGER PART OF}$$

$$= \int_1^{\infty}\frac{x-[x]-\frac{1}{2}}{x}\,dx$$

NUMERATOR IS PERIODIC FUCTION, PERIOD = 1

WANNA INTEGRATE BY PARTS

PAUSE: BERNOULLI POLYNOMIALS

GENERATED BY $\frac{ze^{xz}}{e^z-1} = \sum^{\infty}\frac{B_n(x)}{x}z^n$

SO THAT: $B_n(x) = \sum_{r=0}(2)^{r=0}B_{n-r}\,X^{h}$

$\Rightarrow B_{n-r}^{(0)}$ IS A BERNOUILLI NUMBER

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$$

$$[B(x)]^n = (B+x)^n,\quad B^n = B_n,\quad [B(x)]^n = B_n(x)$$

RECURRSION: $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$

NOTICE THAT $B_1(x) = x - \frac{1}{2}$

BACK TO PROBLEM: $S = \int_1^{\infty}\frac{B_1[x-[x]]}{x}\,dx$

FOR CONVENIENCE, LET $b_n(x) = B(x-[x])$

$\frac{d}{dx}b_n(x) = n\,b_{n-1}(x)$

AND $b_n(1) = B_n(0) = B_n$

AND $b_n(x)$ IS BOUNDED CAUSE

IT IS PERIODIC WITH PERIOD 1

FOR CONVENIENCE IN INTEGRA. BY PARTS

$$b_n(x)/x^n = \frac{1}{n+1}\frac{d}{dx}\left[b_{n+1}(x)/x^n\right] + \frac{n}{n+1}\frac{b_{n+1}(x)}{x^{n+1}}$$

$$S = \int_1^\infty \frac{b_1(x)}{x}\,dx =$$

INTEGRATING BY PARTS

$$S = \frac{b_2(x)}{x}\Big|_1^\infty + \frac{1}{2}\int_1^\infty \frac{b_2(x)}{x^2}\,dx$$

$$= \frac{-B_2}{2} + \frac{1}{2}\left[\frac{1}{3}\frac{b_3(x)}{x^2}\right]_1^\infty + \frac{1}{2}\left(\frac{2}{3}\right)\int_1^\infty \frac{b_3(x)}{x^3}dx$$

$$= -\frac{B_2}{2} + \frac{B_3}{3\cdot 2} + \frac{1}{4\cdot 3}B_4 + \dots$$

$$= -\frac{B_2}{2\cdot 1} - \frac{B_3}{3\cdot 2} - \frac{B_4}{4\cdot 3} - \dots$$

$$= \left[-\frac{B_2}{2\cdot 1} - \frac{B_3}{3\cdot 2} - \dots - \frac{B_{2m}}{2m(2m-1)}\right] + \frac{1}{2m+1}\int_1^\infty \frac{b_{2m+1}(x)}{x^{2m+1}}dx$$

$$B_{2m} \sim (-)^{m+1}\frac{2(2m)!}{(2\pi)^{2m}}$$ ⇐ FROM STERL. FORMU

DEFINE THE FORMAL POWER SERIES:

$$S(x) = \frac{B_2}{2\cdot 1} + \frac{B_3}{3\cdot 2}x + \frac{B_4}{4\cdot 3}x^3 + \dots$$

$$= \sum_{n=0}^{\infty}\frac{B_{n+2}}{(n+2)(n+1)}x^n$$ ⇐ DON'T CONVERGE (RATIO TEST)

AT LEAST, FORMALLY $S(1) = S$

LAPLACE TRANSFORMS:

$S(x) = \sum a_n x^n$ AND $\mathcal{S}(x) = \sum_{n=0} \frac{a_n}{n!}x^n$

$$S(x) = \int_0^\infty e^{-t}\mathcal{S}(tx)\,dt$$

$$\mathcal{S}(x) = \sum_{n=0}^{\infty}\frac{B_{n+2}\,x^n}{(n+2)!}$$ ⇐ CONVERGES

$$= \frac{1}{x^2}\left[-B_0 - B_1(x) + \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n\right]$$

$$= \frac{1}{x^2}\left[-B_0 - B_1 x + \frac{x}{e^x - 1}\right]$$

$$= \frac{1}{x}\left[-\frac{B_0}{x} - B_1 + \frac{1}{e^x - 1}\right]$$

$$= \frac{1}{x}\left[\frac{1}{x} - \frac{1}{2} - \frac{1}{e^x - 1}\right]$$

IN VERSE LAPLACE $\mathcal{S}(tx) = \frac{1}{tx}\left[\frac{1}{2} - \frac{1}{tx} + \frac{1}{e^{tx}-1}\right]$

GIVES $S(x) = -\frac{1}{x}\int_0^\infty\left[\frac{1}{2} - \frac{1}{tx} + \frac{1}{e^{tx}-1}\right]\frac{e^{-t}}{t}dt$

$$r = tx \Rightarrow t = r/x \qquad z = 1/x$$

GIVES $S(x) = -z\int_0^\infty\left[\frac{1}{2} - \frac{1}{r} + \frac{1}{e^r - 1}\right]\frac{e^{-rz}}{r}dr$

LOOK IN A TABLE. GETS YOU

$S(x) = -\frac{1}{x}\ln\Gamma(z) - (z - \frac{1}{2})\ln z + z - \ln\sqrt{2\pi} \Rightarrow S(1) =$

10·28-75

HOMEWORK DO CHAPT. 3, DO 1-20

TRICKS

$$1.\ I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx$$

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = I$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \int_{-\infty}^{\infty}\int e^{-(x^2+y^2)}\,dx\,dy$$

$$\rho^2 = x^2 + y^2 \qquad dx\,dy = \rho\,d\rho\,d\phi$$

$$\Rightarrow I = \int_0^{2\pi}\int_0^{\infty} \rho\, e^{-\rho^2}\,d\rho$$

$$= \pi \int_0^{\infty} e^{-v}\,dv = \pi$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-x_0)^2/2\sigma^2}\,dx = 1$$

$$\int_0^{\infty} e^{-\alpha x^2} = \frac{1}{2}\int_{-\infty} e^{\frac{-x^2}{2\sigma^2}}\,dx \Rightarrow \alpha = \frac{1}{2\sigma^2}$$

$$\Rightarrow = \frac{1}{2}\sqrt{2\pi}\,\sigma = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x\left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)dx = x_0$$

$$\int_{-\infty}^{\infty} (x-x_0)\left(\qquad\right)dx = 0$$

$$\int_{-\infty}^{\infty} (x-x_0)^2\left(\qquad\right)dx = \sigma^2$$

$$= \langle x^2\rangle - \langle x\rangle^2$$

2. $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$

$x = \sqrt{a}$

$\Rightarrow \int_0^\infty e^{-av^2}dv = \frac{1}{2}\sqrt{\pi/a}$

TAKE $\frac{d}{da}$

$\int_0^\infty (v^2)\, e^{-av^2}\, dv \quad -\frac{1}{4}\sqrt{\pi}/a^{3/2}$

$\int_0^\infty v^2 e^{-av^2} dv = \frac{1}{4}\frac{\sqrt{\pi}}{a^{3/2}}$

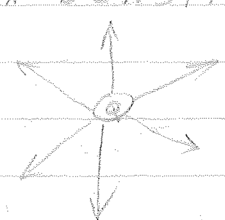
$\int_0^\infty x e^{-ax^2} = \frac{1}{2a}$ ⟵ DIFF. WITH a

11-4-75 (TUES)

$\vec{J}$ = CURRENT DENSITY $\left(\frac{AMP}{M^2} = \frac{COUL}{M^2 SEC}\right)$

"FURR" FLUX = CHARGE FLUX

$\vec{D}$ = FLUX DENSITY (ELECTRIX) $\Rightarrow \frac{COUL}{M^2} = \frac{FURR}{M^2 SEC}$

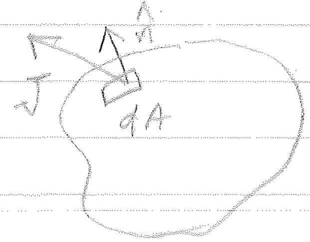


1 coul = 1 $\frac{furr}{sec}$

$F = q[\vec{E} + \vec{v}\times\vec{B}]$

ω FLUX = $\frac{\omega}{mt)^2\, SEC}$

$\oint\!\!\oint \vec{J} \cdot d\vec{s}$



$d\vec{s} = \hat{n}\, dA$

$\iint \vec{J} \cdot d\vec{s} =$ NET (TOTAL AMOUNT OF CHARGE LEAVING V_S

USE DIVERGENCE THEOM

$I = \oint\!\!\oint \vec{J} \cdot d\vec{s} = \iiint \nabla \cdot \vec{J}\, dV$

$\nabla \cdot J + \frac{\delta \rho}{\delta t} = 0 \Leftarrow$ CONTINUITY EQN.

$\Rightarrow I = -\iiint \frac{\delta \rho}{\delta t}\, dV = -\frac{\delta}{\delta t} \iiint \rho\, dV$

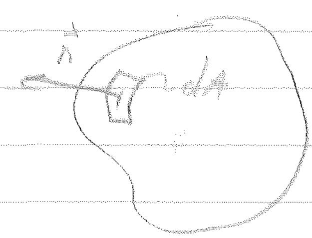
$= -\frac{\delta}{\delta t} Q_T$

ANOTHER WAY: $\oint\!\!\oint \vec{J} \cdot d\vec{s} = -\frac{\delta Q_T}{\delta t}$

TRY FOR $\vec{D}$ FIELD

$\oint\!\!\oint \vec{D} \cdot d\vec{s}$

$d\vec{s} = \hat{n}\, dA$



δq

$\vec{D}$

$\oint\!\!\oint \vec{D} \cdot d\vec{s} = \iiint \nabla \cdot D\, dV \Leftarrow$ FROM DIVERGENCE THEM

$\nabla \cdot D = \rho \Leftarrow$ GAUSS'S LAW (MAXWELL)

$\Rightarrow \iiint \rho\, dV = Q_T = \left(\frac{df}{dt}\right)_T$

$\nabla \cdot D - \rho = 0$, $\left(\text{ie } \nabla \cdot D - \frac{df}{dt} = 0\right)$

COMPARE WITH $\nabla \cdot J + \frac{\delta \rho}{\delta t} = 0$.

2.10. $\frac{(1\cdot3)^2}{(1)(1)(1)^2} + \frac{(1\cdot3\cdot5)}{4\cdot2(1\cdot2)^2} + \ldots$

$$C_n = \frac{(2n+1)!!}{(n!)^2 \cdot n \cdot 4^{n-1}}$$

$$\frac{C_{n+1}}{C_n} = \frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^3}$$

FOR LARGE n

$$\frac{C_{n+1}}{C_n} \simeq \frac{1+\frac{3}{n}}{1+\frac{3}{n}}$$

$$1/1+3/n \simeq 1-\frac{3}{n}$$

$$\frac{C_{n+1}}{C_n} = 1 + O\left(\frac{1}{n^2}\right) \Rightarrow \underline{\text{DIVERGES}}$$

2-12. (SEE ~~Y~~ELLOW NOTES)

$$\sec z = \sum_{n=0} (-)^n \frac{E_{2n}}{(2n)!} z^{2n}$$

ASIDE

COMPARING

① $\nabla\cdot J + \frac{\delta\rho}{\delta t} = 0$ WITH ② $\nabla\cdot D - \frac{\delta f}{\delta t} = 0$

IF THERE WAS A CHARGE "GENERATOR" OR "SOURCE", SAY, ρ_s INSIDE SURFACE, WE'D HAVE

$$\nabla\cdot J + \frac{\delta\rho}{\delta t} - \frac{\delta\rho_s}{\delta t} = 0$$

THE CHARGE IS A FURR SOURCE. i.e, IT GENERATES FURR.

11-6-75 (THURS)

FOR $\vec{J}$

$$\oiint J\cdot ds = \iiint \nabla\cdot J\,dv = \iiint -\frac{\delta\rho}{\delta t}\,dv$$
$$= -\frac{d}{dt}\iiint \rho\,dv = -dQ_T/\delta t$$

FOR $\vec{D}$

$$\oiint D\cdot ds = \iiint \nabla\cdot D\,dv = \iiint \rho\,dv = Q_T$$

$$\frac{1\ \text{FURR}}{\text{SEC}} = 1\ \text{COULOMB} \Rightarrow Q_T = \frac{dF}{dt}$$

CONSIDER $\vec{G}$ AS A FLUX

$$\oiint \vec{G}\cdot d\vec{s} = \iiint \nabla\cdot G\,dV$$

SAY "GARBAGE" FLUX $\Rightarrow$ G HAS UNITS $\frac{\text{GARBAGE}}{\text{SEC } m^2}$

$\vec{G}\cdot d\vec{s}$ HAS UNITS $\frac{\text{GARBAGE}}{\text{SEC}}$ = GARBAGE CURRENT

$\nabla\cdot\vec{G} \Rightarrow \frac{\text{GARBAGE}}{m^3\cdot\text{SEC}}$ = GARBAGE DENSITY/SEC

$\nabla\cdot G\,dV$ = GARBAGE/SEC

SUPPOSE WE IMAGINE THE VOLUME V AS CONTAINED (AT A GIVEN TIME INSTANT) SOME GARBAGE: $\mathcal{G}_{TV}$

$$\frac{\delta\mathcal{G}_T}{\delta t} = -\underbrace{\oiint \vec{G}\cdot ds}_{\substack{\text{AMOUNT}\\\text{LEAVING}\\\text{VOLUME THRU}\\\text{SURFACE}\\\text{PER SECOND}}} + \underbrace{\frac{d\mathcal{G}_{\text{CREATED}}}{dt}}_{\substack{\text{GARBAGE RATE}\\\text{OF "BIRTH"}\\\text{INSIDE VOLUME}}}$$

$$\oiint G\cdot ds = \frac{d\mathcal{G}_{CR}}{\delta t} - \frac{\delta\mathcal{G}_T}{\delta t} = \iiint_V \nabla\cdot G\,dV$$

CASE 1: $G = J \Rightarrow$ GARBAGE = CHARGE

$$\frac{d\mathcal{G}_{CR}}{\delta t} = 0 \Rightarrow \iint J\cdot ds = -\frac{dQ_T}{dt}$$
$$\Rightarrow \nabla\cdot J + \frac{\delta\rho}{\delta t} = 0$$

CASE 2: $G = D$ garbage = FURR

$$\mathcal{G}_{TV} = F_{TV}$$
$$\iiint \nabla\cdot D\,dV = \iiint \rho\,dV = Q_T = \frac{\delta F_{CR}}{\delta t} - \cancel{\frac{\delta F_{TV}}{\delta t}}^{\,0}$$

INTEGRALS:

$$\int_0^u \frac{\sin t}{t}\, dt = Si(u)$$

11-11-75

ANALYTIC FUNCTIONS (pg 477)

$$f(z) = U(z) + i\,V(z)$$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

EX:

$$f(X,Y) = X^2 + Y^2 \Leftarrow \text{NOT ANALYTIC}$$

$$\frac{(X+h_1)^2 + (Y+h_2)^2 - X^2 - Y^2}{h_1 + i\,h_2}$$

$X^2 - Y^2 + 2XYi$ IS ANALYTIC

VERT & HOR. TESTING IS SUFFICIENT!

MAY DERIVE:

$$\frac{\delta U}{\delta X} + i\,\frac{\delta V}{\delta X} = \frac{\delta V}{\delta Y} - i\,\frac{\delta U}{\delta Y}$$

OR

$$\frac{\delta U}{\delta X} = \frac{\delta V}{\delta Y} \quad ; \quad \frac{\delta V}{\delta Y} = -\frac{\delta U}{\delta X} \Leftarrow \text{AT ANY PT}$$

f & g ANALYTIC $\Rightarrow$ $f \pm g$ AN.

$\alpha f \pm \beta g$ AN.

f^n AN, $\frac{f}{g}$ ANALYT

ALL POLY'S ARE ANALYTIC

COS & SIN ANA

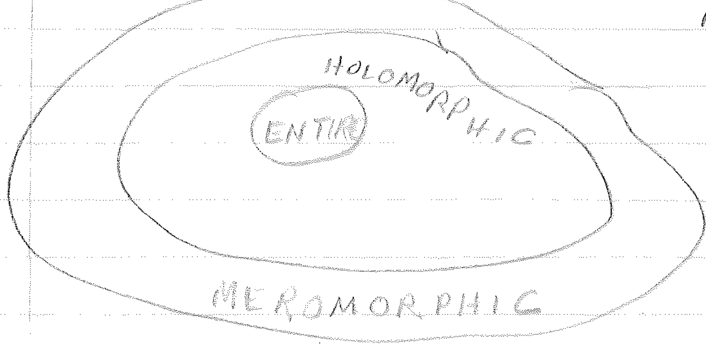
ANALYTIC BUT FOR POLES: HOLOMOMORPHIC

" EVERYWHERE: ENTIRE (REGULAR)

MEROMORPHIC

$\ln z$, $\sqrt{z}$

GOT BRANCH PTS.



$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

INFINATE ORDER POLE = ESSENTIAL SINGULARITY
= "NON-REMOVABLE" SINGULARITY

11-13-75 (THURS)

CAUCHY'S INTEGRAL FORMULA:

$$f(z) = \oint \frac{f(z')}{z-z'}\, dz'$$

$$f(z) = U(x,y) + i\,V(x,y)$$

$$\int_{z_1, C}^{z_2} f(z)\,dz = \int_{z_1, C}^{z_2} (U+iV)(dx+i\,dy)$$

$z_2 = x_2, y_2$; $z_1 = x_1, y_1$

$$= \int_{z_1, C}^{z_2} (U\,dx - V\,dy) + i\,(V\,dx + U\,dy)$$

$$= \int_C (U\,dx + V\,dy) + i\int (V\,dx + U\,dy)$$

$$= \int_C \begin{pmatrix} U \\ V \end{pmatrix} \cdot d\vec{r} + i\int_C \begin{pmatrix} V \\ U \end{pmatrix} \cdot d\vec{r}$$

$\nabla \cdot \vec{B} = 0$

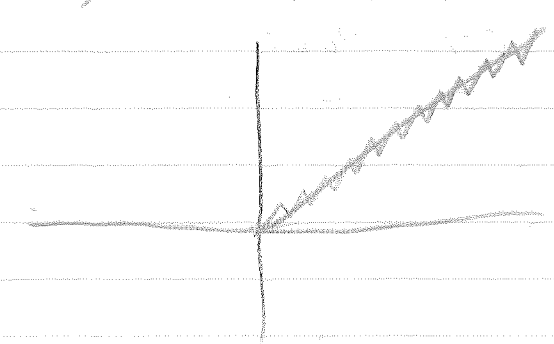
$\nabla \cdot \vec{D} = \rho$

$\nabla \times H = J + \frac{\delta D}{\delta t}$

$\nabla \times E = -\frac{\delta B}{\delta t}$

$$\vec{E} = -\nabla\phi \Rightarrow \phi = \phi_0 - \int_{r_0}^{r} \vec{E} \cdot d\vec{l}$$

FOR $\ln$, USE BRANCH PT: $\ln z = \ln r + i\theta\,(+i2\pi m)$



$$E = -\nabla\phi + \frac{d\vec{A}}{dt}$$

11-20-75 (THURS)

$\vec{E} = -\nabla V \quad \text{IFF } \nabla \times E = 0$

$\nabla \cdot D = \rho \iff \iiint_V \nabla \cdot D \, dv = \iiint_V \rho \, dv \ \forall$ VOLUME

OR $\oiint_S D \cdot ds = Q_V$

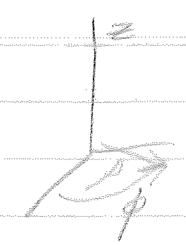
$\Rightarrow \nabla \cdot D = \rho = Q_V / V$

TO GET D OUT OF INTEGRAL, MUST HAVE SYMMETRY. THEN USE GAUSS' LAW

OKAY, $E = -\nabla V$

$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} E(\vec{r}\,') \cdot d\vec{l}$ ⟸ DO NOT HAVE TO DEFINE PATH

EX.



$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 \rho} \hat{e}_\rho = -\nabla V$

V ONLY DEPENDS ON V i.e., $V = V(\rho)$

$V(\rho) = \left(\frac{-\lambda}{2\pi\epsilon_0}\right) \ln \rho + \text{CONST.}$

$= \frac{-\lambda}{2\pi\epsilon_0} \ln \rho/\rho_0$

PRIVATE RESEARCH PROBLEM

COMPARE $\vec{E} = -\nabla V$

$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} E(\vec{r}) \cdot d\vec{l}$

$\vec{E} = -\nabla V = \vec{\nabla} \int_{\vec{r}_0}^{\vec{r}} E(\vec{r}\,') \cdot d\vec{l}$ ⟵ TRY TO SHOW

AKIN TO $\frac{d}{dx} \int_a^x f(x')dx' = f(x)$

COMPLEX VARIABLES:

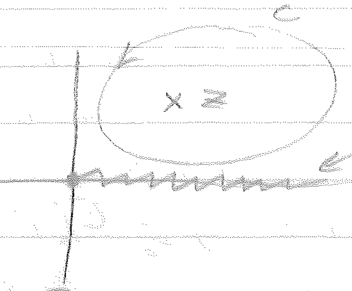
CAUCHY'S INTEGRAL FORMULA

$$f(z) = \frac{1}{2\pi i}\int \frac{f(z')}{z'-z}\,dz$$

IF f IS ANALYTIC <u>INSIDE</u> THE COUNTOUR

NOT ON!

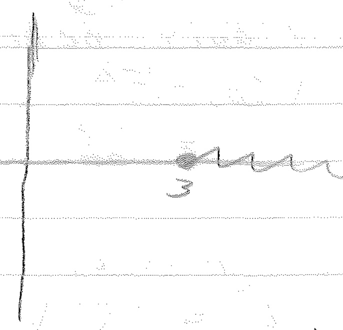
$\oint \frac{\ln z'}{z'-z}\,dz'$



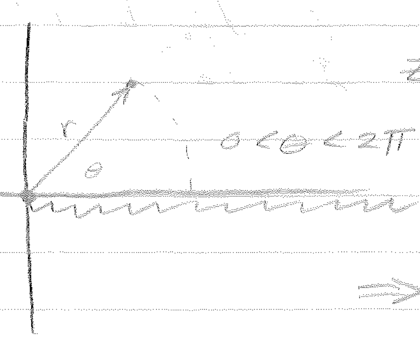
BRANCH DECIDES ON ln function

TRY

$\oint \ln \frac{z-3}{z'-z} \times \frac{1}{2\pi i} = \ln(z-3)$



CONSIDER: $f(z) = \frac{1}{\sqrt{z(z-1)}} = \frac{1}{\sqrt{z}\,\sqrt{z-1}} = \frac{1}{\sqrt{z}} \cdot \frac{1}{\sqrt{z-1}}$



$z^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}\, e^{-i\frac{\theta}{2}}\, \underbrace{e^{-\pi n i}}$

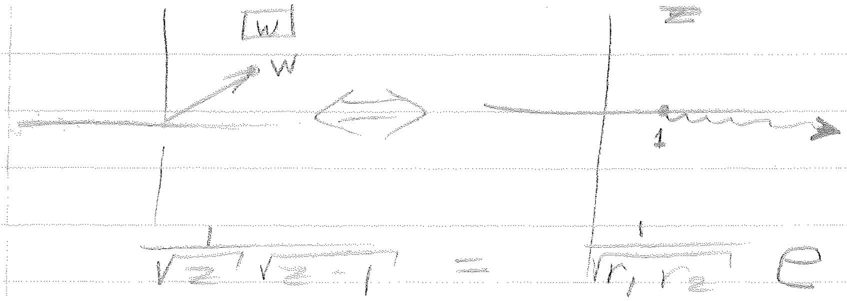
GOTTA CHOOSE n

LET $n = 0$

$\Rightarrow z^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}\, e^{-i\theta/2} = \frac{1}{\sqrt{z}}$

$(z-1)^{-\frac{1}{2}} = w^{-\frac{1}{2}} \qquad \ni w = z-1$

$= |w|\, e^{-1/2\,\theta_w}\, e^{-\pi n i} \Leftrightarrow |z-1|^{-\frac{1}{2}}\, e^{-\frac{i}{2}\theta_2}\, e^{-i\pi n}$



$$\frac{1}{\sqrt{z}\,\sqrt{z-1}} = \frac{1}{\sqrt{r_1 r_2}}\, e^{-\frac{i}{2}(\theta_1+\theta_2)}$$

11-25-75 (TUES)

CUTS:

$f(z) = \frac{1}{\sqrt{z(z-1)}}$

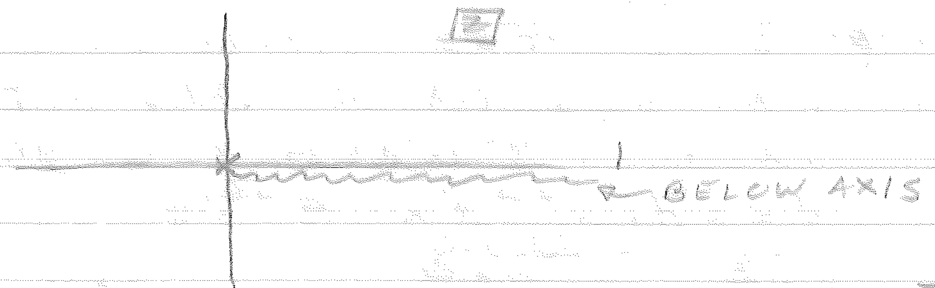
1. MAKE IT A PRODUCT OF FUNCTIONS

$$f(z) = z^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}$$

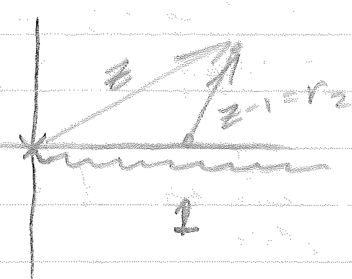
TREAT EACH SEPERATELY

$$z^{-\frac{1}{2}} = (r_1 e^{i\theta} e^{i2\pi n})^{-\frac{1}{2}} = \frac{1}{\sqrt{r_1}} e^{-i\theta_1/2} e^{-i\pi n}$$

$$e^{-i\pi n} = \pm 1 \quad \text{CHOOSE } +1$$

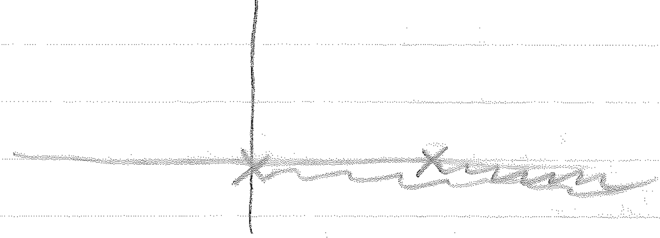


THUS $0 \le \theta_1 < 2\pi \Rightarrow z^{-\frac{1}{2}} = \frac{1}{\sqrt{z}}$



$$\frac{1}{\sqrt{z-1}} = [\underbrace{|z-1|}_{r_2} e^{-i\theta_2} e^{i2\pi m}]^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{r_2}} e^{-i\theta_2/2} e^{i\pi m}$$

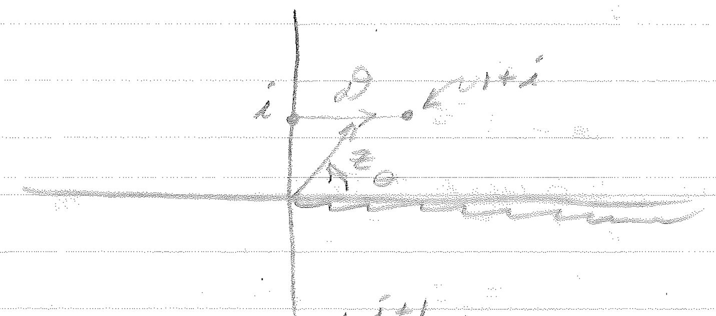


$0 < \theta_1, \theta_2 < 2\pi$

ASIDE:

$$\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{r}} e^{-i\theta/2} \quad ; \; z = re^{i\theta}$$

$$0 \le \theta < 2\pi$$



DOES $I = \int_i^{i+1} \frac{1}{\sqrt{z}} dz$ DEPEND ON $\mathcal{P}$? NO

AS LONG AS $\mathcal{P}$ DON'T GO THRU CUT.

$$r \sin\theta = 1 \Rightarrow dr \sin\theta + r\cos\theta \, d\theta = 0$$

$$dz = dr e^{i\theta} + ire^{i\theta} d\theta$$

$$\frac{dr}{d\theta} = -\frac{r\cos\theta}{\sin\theta}$$

$$\Rightarrow dz = (dr/d\theta + ir) e^{i\theta} d\theta$$

$$dz = \left(-\frac{r\cos\theta}{\sin\theta} + ir\right) e^{i\theta} d\theta$$

$$= \left(-\frac{\cos\theta}{\sin\theta} + i\right) re^{i\theta} d\theta$$

$$= \frac{-r}{\sin\theta} d\theta$$

$$I = \int_{\pi/2}^{\pi/4} \frac{e^{-i\theta/2}}{\sqrt{r}} \left(\frac{-r}{\sin\theta}\right) d\theta$$

$$= -\int_{\pi/2}^{\pi/4} \frac{e^{-i\theta/2}}{(\sin\theta)^{3/2}} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{e^{-i\theta/2}}{(\sin\theta)^{3/2}} d\theta \quad \leftarrow \text{PRETTY HAIRY}$$

TRY SOMETHING ELSE

$$\frac{d\theta}{dr} = -\frac{\sin\theta}{r\cos\theta}$$

$$dz = e^{i\theta}\left(1 - i\frac{\sin\theta}{\cos\theta}\right) dr$$

$$= \frac{dr}{\cos\theta}$$

$$I = \int \frac{1}{\sqrt{r}} e^{-i\theta/2} \frac{dr}{\cos\theta}$$

FROM PATH: $\sin\theta = \frac{1}{r}$, $\cos\theta = \sqrt{1 - \frac{1}{r^2}}$

$$I = \int_1^{\sqrt{2}} \sqrt{\frac{r}{r^2-1}} \, e^{-i\theta/2} dr$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

$$= \sqrt{\frac{1+\sqrt{1-\frac{1}{r^2}}}{2}}$$

$$= \sqrt{\frac{r+\sqrt{r^2-1}}{2r}}$$

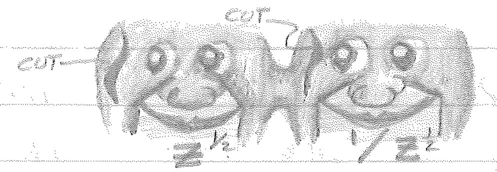
UGLY!

BETTER METHOD

$$I = \int_i^{1+i} \frac{1}{\sqrt{z}}\,dz = 2\sqrt{z}\,\Big|_i^{1+i}$$

BUT WE HAVN'T YET DEFINED $\sqrt{z}$

DEFINE $\sqrt{z}$ TO HAVE SAME CUT AS $\frac{1}{\sqrt{z}}$

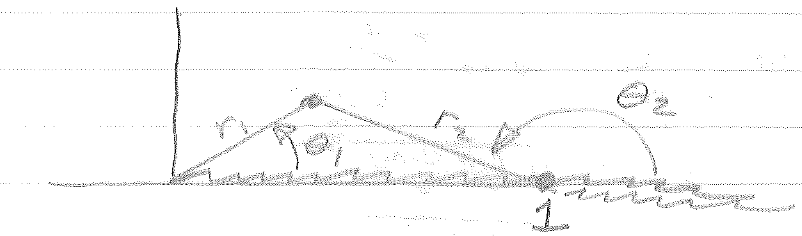


MAY USE ANOTHER CUT IF THE CUT DON GO THRU $\mathcal{D}$.

② STEP 2 (WATCH THAT FIRST STEP, IT'S A DILLY!)

$$\frac{1}{\sqrt{z}\sqrt{z-1}} \overset{①}{=} \frac{1}{\sqrt{z}\ \sqrt{z-1}} \overset{②}{=} \frac{1}{\sqrt{r_1 r_2}}\, e^{-i\frac{(\theta_1+\theta_2)}{2}}$$

$$0 \leq \theta_1, \theta_2 < 2\pi$$

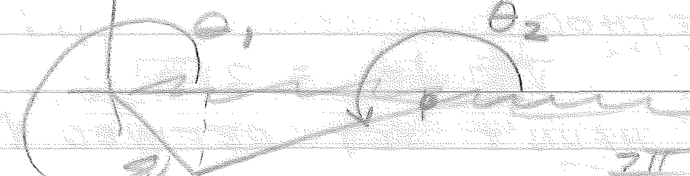


$r_1 \simeq .3 \qquad r_2 \simeq .8$

$\theta_1 \simeq 45^\circ = \frac{\pi}{4} \qquad \theta_2 \simeq \frac{3\pi}{4}$

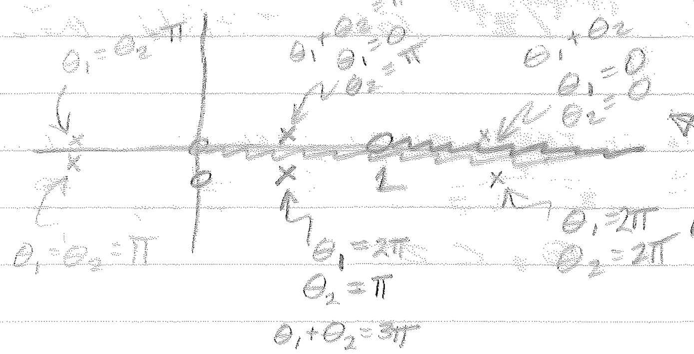
$\frac{1}{2}(\theta_1 + \theta_2) = \frac{\pi}{2}$

$\Rightarrow \frac{1}{\sqrt{r_1 r_2}} e^{-i\left(\frac{\theta_1+\theta_2}{2}\right)} = \frac{-i}{\sqrt{(.3)(.8)}}$



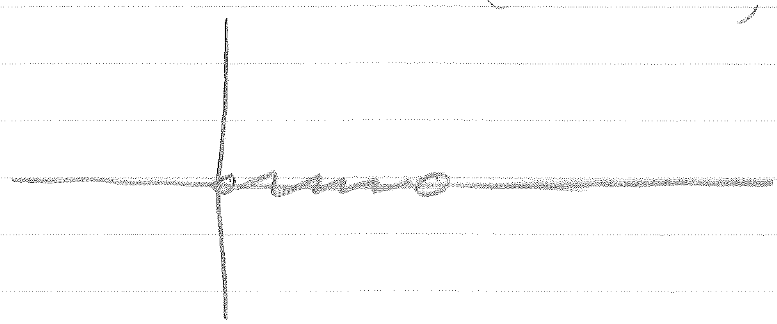
$\theta_1 = \frac{7\pi}{4}$

CUT DISCONTINUITY; WE ARE INTERESTED IN ~~$\theta_1 + \theta_2$~~ OR $e^{-i(\theta_1+\theta_2)}$



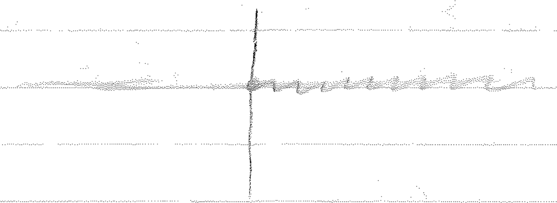
UNDETECTED BY $e^{-i(\theta_1+\theta_2)}$

DISC. $f(z) = \begin{cases} 0 & ; x < 0 \\ 2f(x+\epsilon i); & 0 \le x \le 1 \\ 0 & ; x > 1 \end{cases}$



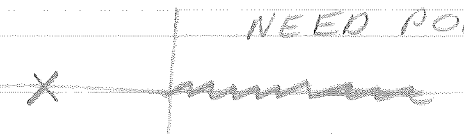
TWO SQUARE ROOTS CAN EAT EACH OTHER

GIVEN CUT AND DISCONTINUITY



$\sqrt{z} = 2\sqrt{|z|}$

Z ABOVE − Z BELOW.



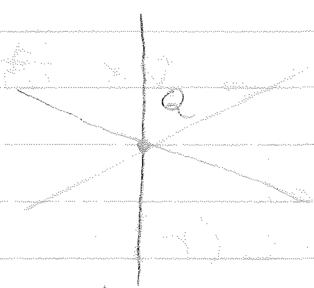
NEED POLES & RESIDUE

$\sqrt{z} + \frac{RES}{z+1}$

YOU CAN ADD, THOUGH, ANY ANALYTIC FUNCTION. CAN RESTRICT BY SAYING $f(z)$ LOOKS LIKE $\sqrt{z}$ @ ∞.

12-2-75 (MON)

A PONDER PROBLEM



$$\vec{D} = \frac{Q}{4\pi r^2}\hat{e}_r$$

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{e}_r$$

$$\vec{D} = \epsilon\vec{E}$$

MAXWELL'S

$$\left\{\begin{array}{ll} \nabla\cdot\vec{D} = \rho & \vec{B} = \mu\vec{H} \\ \nabla\cdot\vec{B} = 0 & \vec{D} = \epsilon\vec{E} \\ \nabla\times\vec{E} = -\frac{\delta\vec{B}}{\delta t} & \vec{J} = \sigma\vec{E} \\ \nabla\times\vec{H} = \vec{J} + \frac{\delta\vec{D}}{\delta t} & \end{array}\right\}$$

$$\vec{S} = \vec{E}\times\vec{H}$$

$$\vec{F} = Q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

$$\vec{F} = m\vec{a}$$

POINT SOURCE : $\rho = Q\,\delta(\vec{r})$

$$\Rightarrow \nabla\cdot\vec{D} = Q\,\delta\vec{r} \ \text{ GIVES } \ \vec{D} = \frac{Q}{4\pi r^2}\hat{e}_r$$

LET $\epsilon = \epsilon_0[2+\cos\theta]\,(1+\frac{b}{a}e^{-r/a})$

$\vec{B}=0 \quad \vec{H}=0$

$$\left.\begin{array}{l}\nabla\cdot D=\rho\\ B=0\\ H=0\\ \nabla\times\vec{E}=0\\ J=-\frac{\delta D}{\delta t}\end{array}\right\}$$

ELECTROSTATIC $\Rightarrow J=0 \Rightarrow \frac{\delta\vec{D}}{\delta t}=0$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{e}_r \qquad \vec{\nabla}\cdot\vec{D} = \rho\,\delta(\vec{r})$$

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{e}_r \Leftarrow \text{NO! } \nabla\times E\neq 0!$$

$$\nabla\times\vec{E} = \nabla\times\left[\frac{1}{\epsilon}\left(\frac{Q}{4\pi r^2}\hat{e}_r\right)\right]$$

$$= \left(\nabla\frac{1}{\epsilon}\right)\times\left(\frac{Q}{4\pi r^2}\hat{e}_r\right) + \frac{1}{\epsilon}\cancel{\nabla\times\frac{Q}{4\pi r^2}\hat{e}_r} \to 0$$

$$= \left(\nabla\frac{1}{\epsilon}\right)\times\left(\frac{Q}{4\pi r^2}\hat{e}_r\right)$$

$\neq 0$ ONLY IF $\epsilon = \epsilon(r)$

WRONG FIELD

$$\iiint\nabla\cdot\vec{D}\,dv = \iiint\rho\,dv = Q = \oiint\vec{D}\cdot ds$$
$$= |D|\iint ds$$

$$= |D|\,4\pi r^2 = Q \Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{e}_r$$

WRONG!

ONE METHOD

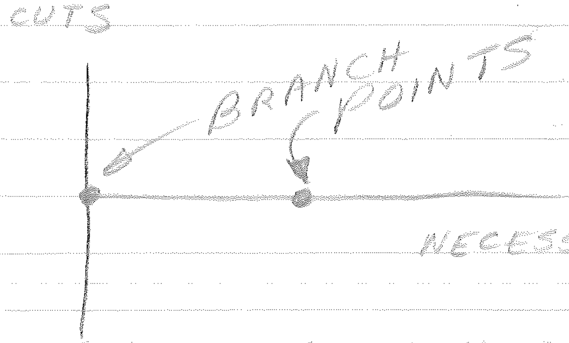
$$\nabla\cdot(\epsilon\vec{E}) = Q\,\delta(\vec{r})\ ;\ \vec{\nabla}\times\vec{E}=0$$

SOLVE TWO D.E. EQUATIONS

(TRY TO FIND A BETTER WAY)

$$-\nabla\phi = \vec{E} \Rightarrow \vec{\nabla}\cdot[\epsilon\vec{\nabla}\phi] = Q\,\delta(\vec{r})$$

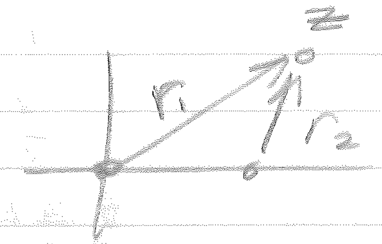
CUTS



$$f(z) = z^{\alpha}(z-1)^{\beta}$$

NECESSARY FOR $\alpha, \beta \notin$ INTEGERS

MAKE IT A FUNCTION:



$$f(z) = z^{\alpha}(z-1)^{\beta}$$

$$= r_1^{\alpha} r_2^{\beta} \left[e^{i\theta_1}\right]^{\alpha} \left[e^{i\theta_2}\right]^{\beta}$$

$$= r_1^{\alpha} r_2^{\beta} \left[e^{i\theta_1} e^{i2\pi n}\right]^{\alpha} \left[e^{i\theta_2} e^{i2\pi m}\right]^{\beta}$$

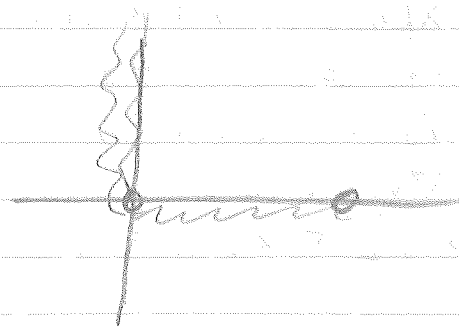
m AND n MUST BE INTEGERS

$$f(z) = r_1^{\alpha} r_2^{\beta} e^{i\alpha\theta_1} e^{i\beta\theta_2} e^{2\pi i(\alpha n + \beta m)}$$

CHOOSE $m = n = 0$

$$\Rightarrow f(z) = r_1^{\alpha} r_2^{\beta} e^{i(\alpha\theta_1 + \beta\theta_2)}$$

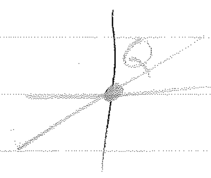
MUST SPECIFY HOW TO MEASURE θ_1 AND θ_2. PUT IN CUTS. THIS MAKES $f(z)$ A FUNCTION



$$-\pi < \theta_1 \leq \pi$$

SOLN' IS: IF $(\alpha + \beta) \in$ INT, CUTS EAT EACH OTHER

12-4-74 (THURS)



$$\vec{D} = \epsilon \vec{E}$$
$$\nabla \cdot D = Q\,\delta(\vec{r})$$

$$\nabla \cdot D = Q\,\delta(\vec{r})$$
$$B = 0$$
$$H = 0$$
$$\underline{J} = 0$$
$$\nabla \times \underline{E} = 0$$
$$\nabla \cdot (\epsilon \vec{E}) = Q\,\delta(\vec{r})$$
$$\vec{E} = -\nabla V \quad \Longrightarrow \nabla \cdot (\epsilon \nabla V) = -Q\,\delta(\vec{r})$$

ANOTHER WAY: $\vec{D} = \frac{Q}{4\pi r^2}\hat{e}_r + \underbrace{\nabla \times \vec{G}}_{\vec{K}}$

NOW MUST SATISFY $\vec{\nabla} \times \vec{E} = 0$

NOW: $\vec{E} = \text{(NO CURL)} + \text{(NO DIVERG)}$

$$= -\vec{\nabla} V + \vec{\nabla} \times \vec{\Pi}$$
$$\vec{E} = \frac{1}{\epsilon}\frac{Q}{4\pi r^2}\hat{e}_r + \frac{1}{\epsilon}\vec{K}$$
$$\vec{\nabla} \times \vec{E} = 0 = \left(\nabla \frac{1}{\epsilon}\right) \times \left[\frac{Q}{4\pi r^2}\hat{e}_r + \vec{K}\right] + \frac{1}{\epsilon}\nabla \times \vec{K}\,; \quad \vec{\nabla}\cdot\vec{K} = 0$$

GIVEN ϵ, AND TRY TO GUESS SOLUTION

BACK TO $\nabla \cdot (\epsilon \nabla V) = -Q\,\delta(\vec{r})$

SAYS $\epsilon \nabla V = \nabla \times J$

$$\nabla \times \left[\frac{1}{\epsilon}\nabla \times J\right] = 0$$

ANYWAY, CONCLUDE D FIELD DEPENDS ON ϵ IN THE GENERAL TREATMENT.

CHAPTER 7

SPECIAL (HIGHER TRANSCENDENTAL) FUNCTIONS

LOWER TRANSCEN. FUNCTIONS: ln, cos, sin

cos, sin log, Legendre, Bessel, Hermite, Whittaker,
confluent hypergeometric, mathieu,
gamma, digamma, tri ...,
eponential integral, sine & cos integrals,
fresnel integrals, elliptic function, erf

LEGENDRE FUNCTIONS

$P_\ell^m(\cos\theta)$ $Q_\ell^m(\cos\theta)$

BESSEL J_ℓ Y_ℓ N_ℓ ⇐ ORDYNARY

K_ℓ I_ℓ ⇐ MODIFIED

$\ell = \pm\frac{n}{2}$ ⇐ SPHERICAL

HYPERGEOMETRIC

${}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$

CONF: ${}_1F_1(a; b; x)$

GENERAL THINGS WE'D LIKE TO KNOW

1. DIFFERENTIAL EQATION
   USUALLY SECOND ORDER: $G'' = f(G, G')$
2. POWER SERIES EXPANSION
3. RECURRSION RELATIONS (DIFFERENCE EQUA)
4. INTEGRAL REPRESENTATIONS
5. INTEGRAL EQUATIONS
6. GENERATING FUNCTION
7. CONNECTION WITH OTHER TRANSCENDENTAL FUNCTIONS
8. SPECIAL CASES
9. INTEGRAL & DERIVATIVES
10. ORTHOGONAL ?

12-10-75 (WED)

DIFFEREN. EQ: (SECOND ORDER LINEAR

$$Y'' + f(x)Y' + g(x)Y = h(x)$$

$$\hat{D}\,Y = h \qquad \hat{D}\,Y_p(x) = h(x)$$

$$\hat{D}\,Y_H(x) = 0$$

$$Y = Y_p + AY_{H_1} + BY_{H_2}$$

CONSIDER THEN

$$Y'' + f(x)Y' + g(x)Y = 0$$

Y_{H_1} AND Y_H IND $\Rightarrow Y_{H_1} \neq CY_{H_2}$

Y_1, Y_2, Y_3 IND $\Rightarrow AY_1 + BY_2 + CY_3 \neq 0$

CHECK FOR LINEAR INDEPENDENCE

WRONSKIAN: $fg' - f'g$

$$= \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = 0 \Leftarrow \text{NOT INDENDENT}$$

IS TESTING IF

$$\frac{f'}{f} = \frac{g'}{g}$$

$$\frac{d}{dx}\ln f = \frac{d}{dx}\ln g \Rightarrow \frac{d}{dx}\ln\frac{f}{g} = 0$$

EXAMPLE: $j_0(x) \sim \frac{1}{\sqrt{x}} J_{1/2}(x) \sim \frac{\sin x}{x}$

MAY TEST IF $j_0(x)$ IS $\sim \sin(x)$

WRONSKIAN OF TWO BESSEL FUNCTION

$$n_\ell j_\ell' - n_\ell' j_\ell = W(j_\ell, n_\ell)$$

MAY GET FROM INSPECTION OF ORIGINAL D.E.

CONSIDER GEN LIN. D.E

y, z $\Leftarrow$ SOLUTIONS

$$z\,y'' + f\,z\,y' + g\,z\,y = 0$$

$$y\,z'' + f\,y\,z' + g\,z\,y$$

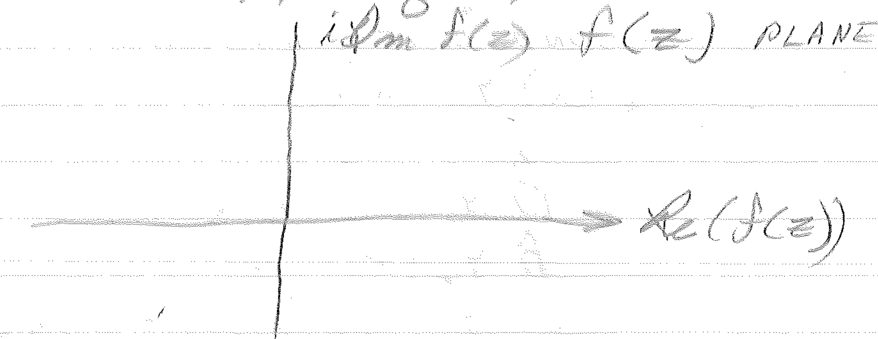
SUBTRACT

$$[z y'' - y z''] + f(z y' - y z') = 0$$

$$\Rightarrow W' + fW = 0 \Leftarrow \text{FIRST ORDER DE}$$

W = WRONSKIAN $\Rightarrow \frac{d \log W}{dx} = f(x)$

$Y'' + f(x)Y' + g(x)Y = 0$



$f(z), g(z)$ IS COMPLEX GENERALIZATION.

SPECIAL CASE:

$Y'' + \frac{2}{X}Y' = 0$

$f(x) = \frac{2}{x}$

$f(z) = \frac{2}{z}$ ← GOTTA POLE

MAKE ASSUMPTION: ALL Y HAS POLE @ ORIGIN

PLUG IN $Y = X^{\sigma}$

GET $\sigma^2 + \sigma = 0$; $\sigma = 0$, $\sigma = -1$

$\Rightarrow Y = C$, $Y = C/X$

NOTE, ALL SOLUTIONS AREN'T SICK @ ORIGIN. $Y = C$ ISN'T.

GENERAL SOLUTION:

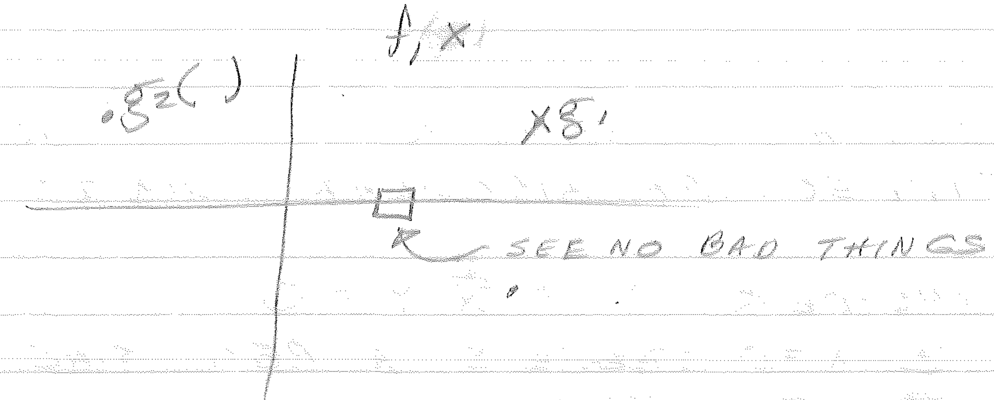
$Y = C_1 + C_2/x$

LINEAR INDEPENDENT

EXAMPLE: $g(z) = \dfrac{G(z)}{(z-q_1)^{\alpha_1}(z-q_1)^{\alpha_2}\cdots(z-q_n)^{\alpha_n}}$

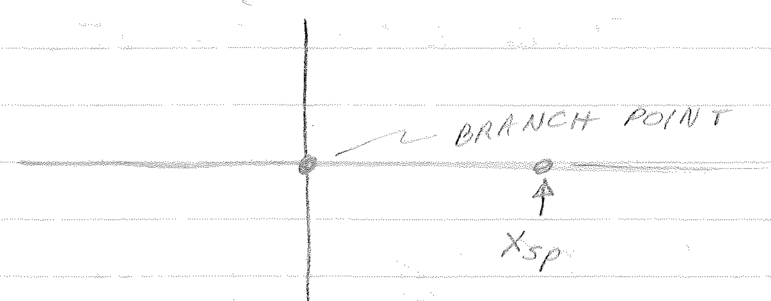
$f(z) = \dfrac{F(z)}{(z-f_1)\cdots(z-}$

α_i = POSITIVE INTEGERS



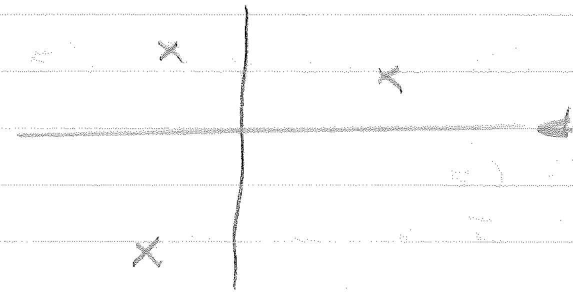
A "REGULAR POINT" OF DIFFER EQN, $f(x)$, $\frac{f}{g}(x)$ IS FINITE & NO BR. POINT

CONSIDER $Y'' + \ln x\ Y$



REGULARITY DETERMINED BY CUT

$\Rightarrow$ A "REGULAR POINT" OF A D.E. IS A POINT @ WHICH ALL COEFFICIENTS ARE ANALYTIC.



HERE ON REAL AXIS, ALL POINTS ARE ANALYTIC.

① A "REGULAR SINGULAR POINT" IS

② AN "IRREGULAR SINGULAR POINT"

CONSIDER $Y'' + \frac{2}{x} Y = 0$

@ $x = 0$ THERE'S A REG SING POINT

② I.S.P. ARE UNINTERESTING

AN ESSENTIAL SINGULAR PT. OF THE D.E. IS A POINT WHICH IS NEITHER A REGULAR POINT NOR A REGULAR SINGULAR POINT.

CONSIDER

$$Y'' + f(x)\, Y' + g(x)\, Y = 0$$

BUT x_0 IS A REGULAR POINT OF THE D.E.

(i.e. f & g ARE REGULAR POINTS OF f & g)

LET $\gamma = x - x_0$

$$\Rightarrow \frac{dY}{dx} = \frac{dY}{d\gamma} \qquad Y(x) = \tilde{Y}(\gamma)$$

$$d^2Y/dx^2 = d^2Y/d\gamma^2 \Rightarrow f(x) = f(\gamma + x_0) = \tilde{f}(\gamma)$$

$$\tilde{Y}'' + \tilde{f}\, \tilde{Y}' + \tilde{g}\, Y = 0$$

IS ORIGIN A REGULAR OR REGULAR SINGULAR POINT?

ASSUME

THE ORIGIN IS A REGULAR POINT OF THE D.E. IF BOTH f & g ARE ANALYTIC @ A POINT, SO IS SOLUTION. Y, THUS, WE CAN WRITE DOWN TAYLOR SERIES. LET

$$Y = \sum_n A_n X^n$$

FOR SECOND ORDER, ONLY 2 D.O.F. FOR A

TIE DOWN A_0 A_1

THEN $A_2 = F(A_0, A_1)$ ...

SPECIAL CASE:

$$Y'' + 3Y' + XY = 0$$

PLUG AWAY

$$Y'' = \sum_{n=0}^{\infty} A_n n(n-1) X^{n-2} \; ; Y' = \sum n A_n X^{n-1}$$

GIVES

$$\sum_{n=0} A_n n(n-1) X^{n-1} + \sum_{n=0}^{\infty} 3n A_n X^{n-1} + \sum_{n=0}^{\infty} A_n X^{n+1} = 0$$

$$+3A_1 + \underbrace{\sum_{n=2}^{\infty} 3n A_n X^{n-1}}_{n-1=m+1}$$

$$\underbrace{2A_2 + \sum_{n=3}^{\infty} A_n n(n-1) \cancel{X}^{n-2}}_{n-2=m+1} + \text{SAME} = 0$$

$$2A_2 + \sum_{m=0}^{\infty} A_{m+3}(m+3)(m+2) X^{m+1}$$

$$+3A_1 + \sum_{m=0}^{\infty} 3(m+2) A_{m+2} X^{m+1}$$

$$+ \sum_{n=0}^{\infty} A_n X^{n+1} = 0$$

GROUP EQUAL POWERS OF X

$$2A_2+3A_1+\sum_{n=0}^{\infty}\left[A_n+(n+2)(n+3)A_{n+3}+3(n+2)A_{n+2}\right]X^{n+1}=0$$

MAY LOOK AT AS TAYLOR EXPANSION OF $0=\sum A_p X^p \Rightarrow A_p=0$

$\therefore$ ① $2A_2+3A_1=0$

② $A_n+(n+2)(n+3)A_{n+3}+3(n+2)A_{n+2}=0$

$n=0,1,2,3,\ldots$

APPROXIMATE:

$$Y=A_0+A_1X+A_2X^2+A_3X^3$$

$\boxed{\begin{matrix}A_0\\A_1\end{matrix}} \leftarrow$ THE TWO D.O.F. (D.E. CONSTANTS)

NOW: $A_2=-\frac{3}{2}A_1$

$A_2 = -\frac{3}{2}A_1$

$A_0+2\cdot3\cdot A_3+3(2)A_2=0$

$A_3=-A_2-A_0/6=\frac{-A_0}{6}+\frac{3}{2}A_1$

$$Y\simeq A_0+A_1X-\frac{3}{2}A_1X^2-\left(\frac{A_0}{6}-\frac{3}{2}A_1\right)X^3+H.O.T$$

ANSWER GOOD NEAR $X\simeq0$

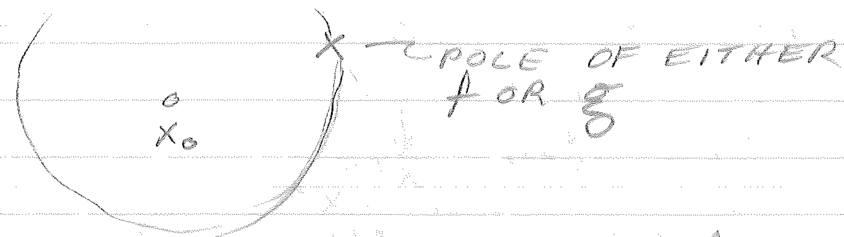
HOW 'BOUT

$f(x)=\log(x+3)$

EXPAND IN POWER SERIES

$\log(x+3)=\ln 3+\frac{1}{3}X-\frac{1}{18}X^2+\ldots$

WHAT ABOUT CONVERGENCE. WILL WORK 'TILL YOU HIT A POLE



HOW 'BOUT CLOSED FORM FOR A_n.
RECALL DIFFERENCE EQ.

$$A_n + (n+3)(n+3)A_{n+3} + 3(n+2)A_{n+1}$$

CONSIDER "WISH AWAY" SOLN

$$A_n + 3(n+2)A_{n+2} = 0$$

$$A_{n+2} = \frac{-1}{3(n+2)} A_n \Leftarrow$$ 2nd ORDER DIFFERENCE EQ. (FIRST ORDER TERM GONE)

NOTE $A_{2m} \Leftarrow A_0$
$A_{2m+1} \Leftarrow A_1$

FOR A_0

$$A_2 = -\frac{1}{3(2)} A_0$$
$$A_4 = -\frac{1}{3(4)} A_2 = \frac{-1}{3(4)3(2)}A_0$$
$$A_6 = -\frac{1}{3\cdot 6} A_4 = \frac{-1}{3(6)3(4)3(2)} A_0$$
$$A_8 = -\frac{1}{3\cdot 8} A_6 = \frac{1}{3(8)3(4)3(2)} A_0$$

$$A_2 = \left(-\frac{1}{3}\right)^1 \frac{1}{2!!} A_0$$
$$A_4 = \left(-\frac{1}{3}\right)^2 1/4!! A_0$$
$$A_6 = \left(-\frac{1}{3}\right)^3 1/6!! A_0$$
$$A_8 = \left(-\frac{1}{3}\right)^4 1/8!! A_0$$

$$A_{2n} = \left(-\frac{1}{3}\right)^n \frac{1}{(2n)!!} A_0$$

$$\Rightarrow Y = A_0 \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{1}{(2n)!!} X^{2n}$$
$$+ A_1 \sum_{n=0}^{\infty} (\quad)^{2n+1}$$

THREE TERM SOLUTION IS TO HAIRY.

METHOD: PROBE WITH $X^n = Y$

$Y'' + 3Y' + XY = 0$

$n(n-1)X^{n-2} + 3nX^{n-1} + X^{n+1} = 0$

$Y'' + 3Y' + XY = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$X^{n+2} \quad X^{n+1} \quad X^{n-1}$

GET'S 'EM ALL X^n

$A_{n+2}X^{n+2},\ A_{n+1}X^{n+1},\ A_{n-1}X^{n-1}$

GIVES

$X^n\left[A_{n+2}(n+2)(n+1) + 3A_{n+1}(n+1) + A_{n-1}\right] = 0$

DOESN'T WORK FOR A_0

REDO FOR SMALL n

TRY $n=0$ GIVES $2A_2 + 3A_1 = 0$

REGULAR SINGULAR POINT EX.

$Y''' + f_1(x)Y'' + f_2(x)Y' + f_3(x)Y = 0$

NOTE: MUST ISOLATE HIGH TERM TO CHECK FOR REGULAR POINTS.

1. ALLOW AT MOST, A FIRST ORDER POLE AT THE ORIGIN IN f_1
2. ALLOW AT MOST, A SECOND ORDER POLE @ ORIGIN IN f_2
3. ALLOW @ MOST A THIRD ORDER POLE @ ORIGIN FOR f_3

HIGHER ORDER D.E.'S APPLICATION OBVIOUS.

IF WORSE THE ABOVE BOUND, ITS AN ESSENTIAL SINGULARITY. GO HOME.

EXAMPLE: $Y'' + \frac{3}{X(X-1)} Y' + \frac{7}{X^2} Y = 0$

ORIGIN IS A REGULAR SINGULAR POINT

X=1 " " " " "

TRY XPANSION OF THE FORM

$$Y = \underbrace{Z^{\sigma}}_{\text{NOT ANA}} \underbrace{\sum_{n=0}^{\infty} A_n Z^n}_{\text{ANA}}, \quad A_0 \neq 0 \Leftarrow \text{GENERALIZED TAYLOR SERIES}$$

EXAMPLE (REG. SING. POINT)

$$Y'' + \frac{3}{X} Y' + Y = 0$$

WANNA OUTPUT $Z^{\sigma+n}$

$$A_{n+2} Z^{\sigma+n+2}, \quad A_{n+2} Z^{\sigma+n+2}, \quad A_n Z^{\sigma+n}$$

$$Z^{\sigma+n}\left[A_{n+2}(\sigma+n+2)(\sigma+n+1) + 3(\sigma+n+2)A_{n+2} + A_n\right] = 0$$

$$A_{n+2}(\sigma+n+2)\left[(\sigma+n+1)+3\right] + A_n = 0$$

$$A_{n+2} = \frac{-A_n}{(\sigma+n+2)(\sigma+n+4)}$$

MUST INPUT $A_0 \neq 0$ AND A_1

MUST CHECK LOWER ORDER TERMS.

A_0 & A_1 ARE INPUTS. CHECK A_3 & MAYBE A_4

$$Y'' + \frac{3}{X} Y' + Y = 0$$

$n=0$: $A_2\left[(\sigma+2)(\sigma+1) + 3(\sigma+2)\right] + A_0 = 0 \Leftarrow Z^{\sigma+2}$

$$\Rightarrow A_2 = -\frac{A_0}{(\sigma+2)(\sigma+4)} \Leftarrow \text{FITS GENERAL PATTERN}$$

PLUG IN $Y = A_1 Z^{\sigma+1}$ NOTHING CAN GO HERE!

$$Y'' + \frac{3}{X} Y' + Y = 0$$

$$\Rightarrow Z^{\sigma-1}\left[(\sigma+1)\sigma + 3(\sigma+1)\right]A_1 = 0$$

AN INPUT

PLUG IN $Y = A_0 Z^{\sigma}$

$$Z^{\sigma-2}\left[\sigma(\sigma-1) + 3\sigma\right]A_0 = 0$$

MAY NOW FIND σ AS

THUS:

$$\sigma(\sigma+2)=0$$

$$(\sigma+1)(\sigma+3)=0$$

MUST USE $\sigma=0$, $\sigma=-2$

THUS, WE **MUST** HAVE $A_1=0$

WE GOT SOLUTION

$$\begin{cases} Y_1 = Z^0 \sum_{n=0}^{\infty} A_n Z^n \quad ; A_0 \neq 0 \\ Y_2 = Z^{-2} \sum_{n=0}^{\infty} A_n Z^n \quad ; A_0 \neq 0 \end{cases}$$

Y_1 HAS: $A_{n+2} = \dfrac{-A_n}{(n+2)(n+4)}$

Y_2 HAS: $A_{n+2} = \dfrac{-A_n}{n(n+2)}$

TWO INDEPENDENT SOLUTIONS.

NOTE: BOTH ARE EVEN FUNCTIONS

($i.e.$ $A_{2p+1} = 0 \ \forall p$)

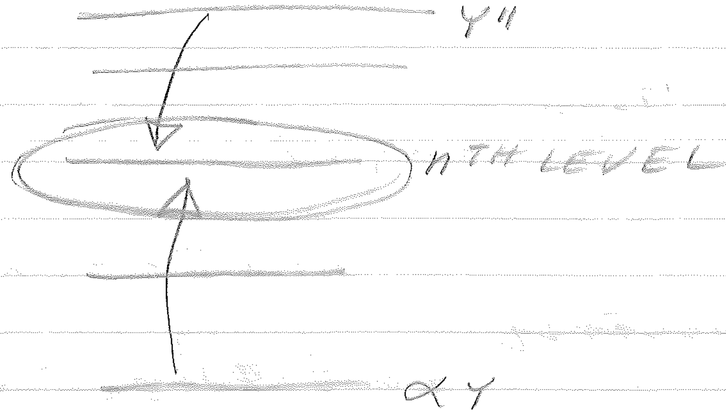
PARITY

$Y'' + X^4 Y = 0$ ← EITHER EVEN OR ODD Y

$Y'' + X^3 Y' + Y = 0$ " " " " "

A TRICK TO REDUCE A 3^{RD} ORDER DIFFERENCE EQ. TO A 2^{nD} ORDER

$$Y'' + (x^2 + \alpha) Y = 0 \Leftarrow \text{WILL GIVE 3 TERM RECURRSION}$$



TRY TO FACTOR SOMETHING OUT OF THE TAYLOR SERIES.

$$Y'' + (x^2 + \alpha) Y = 0 \Leftarrow \text{SINGULAR @ } \infty$$

$$@\infty: \quad Y'' + X^2 Y = 0 \Rightarrow Y_\infty = e^{\pm \frac{1}{2} x^2}$$

SO TRY

$$Y = e^{-\frac{1}{2} X^2} \sum_{n=0}^{\infty} A_n X^n$$

CALLED: "FACTORING OUT THE BEHAVIOR AT A SINGULAR POINT"

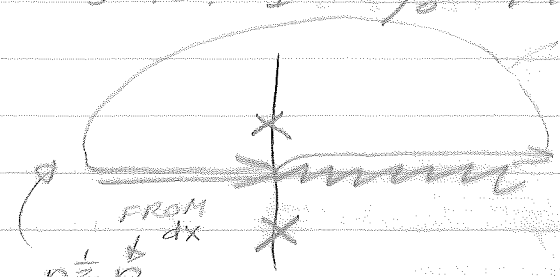
12-11-75 (THURS)

EXAM: OPEN BOOK, 4 PROBLEMS

NO CALCULATORS OR INTEGRAL TABLES

COUNTOUR INT.

3-40: $I = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$



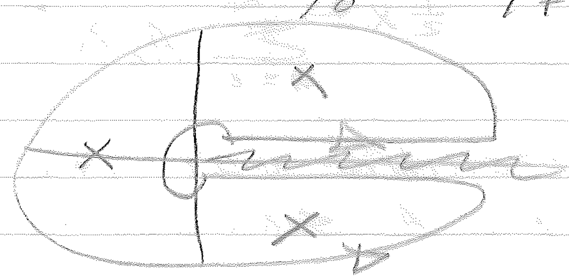
$I + iI = 2\pi i R$

$\operatorname{Res} f(z)\Big|_{z=i} = \frac{\sqrt{i}}{2i} = \frac{1+i}{2\sqrt{2}\,i}$

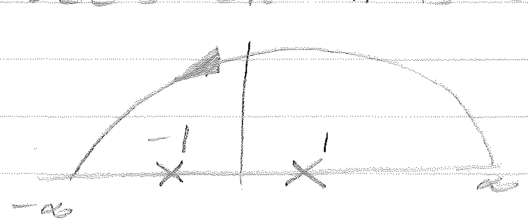
$\lim_{R\to\infty} \frac{R^{\frac{1}{2}} R}{R^2} = 0 \qquad \Rightarrow I = \pi/\sqrt{2}$

EX: $\int_0^\infty \frac{dz}{1+z^2}\,dz$

CONSIDER $\int_0^\infty \frac{\ln z \; dz}{1+z^2}$



POLES ON AXIS OF INTEGRATION



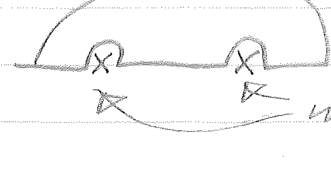
$Y'' - Y = 6e^{-x^2}$

$(4k^2+1)\tilde{Y} = \alpha e^{-\beta k^2}$

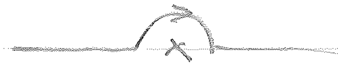
$\tilde{Y} = \frac{\alpha e^{-\beta k^2}}{k^2 - 1}$

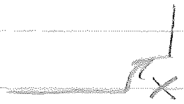
UNIQUE, BUT D.E. EQ.

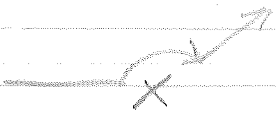
$Ae^{-ix} + Be^{ix}$ ADD ON ← SAYS GOT 2 IND SOL.



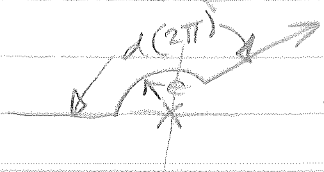
WILL GIVE HOMOGENEOUS SOLUTION

 GIVES $\to -i\pi R$

 GIVES $\frac{i2\pi R}{4}$

 GIVES $i\,2\pi R/3$

CONSIDER



$$I = \int_{\text{REAL } \epsilon} \frac{e^{-z}}{z}\,dz = \int_{\pi}^{\pi-\alpha 2\pi} \frac{e^{-\epsilon e^{i\theta}}}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta$$

$$= i\int_{\pi}^{\pi-\alpha(2\pi)} e^{-\epsilon e^{i\theta}}\,d\theta$$

$$\lim_{\epsilon\to 0} I = i\int_{\pi}^{\pi-\alpha(2\pi)} \left((1-\epsilon e^{i\theta}) + \text{HOT}\right)d\theta$$

$$= i\int_{\pi}^{\pi-\alpha(2\pi)}$$

$$= -i(2\pi)\alpha$$

FINAL ON

TUES: 2-8

MARKS EE 5371
DUE: 9-11-75

BOB MARKS

1.1. $X^2Y' + Y^2 = XYY'$

$Y'(X^2 - XY) + Y^2 = 0$

$X(X-Y)\,dY + Y^2 dX = 0$

LET $Y = vX \Rightarrow dY = X\,dv + v\,dX$ ⇐ VARIABLE SUB.

$X(X - vX)(X\,dv + v\,dX) + v^2X^2 dX = 0$

$X^3(1-v)\,dv + (X^2v - v^2X^2 + v^2X^2)\,dX = 0$

$X^3(1-v)\,dv + X^2 v\,dX = 0$

$\frac{1-v}{v}dv + \frac{1}{X}dX = 0$

$(\frac{1}{v} - 1)\,dv + \frac{1}{X}dX = 0$ ⇐ SEPERATED

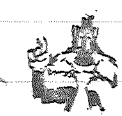
$\ln v - v + \ln X = C_1$

$\ln \frac{Y}{X} + \ln X = \frac{Y}{X} + C_1$

$\ln Y = \frac{Y}{X} + C_1$

$Y = e^{Y/X + C_1}$

$\therefore Y = Ce^{Y/X}$ ⇐ ANSWER



1-2. $Y' = \dfrac{X\sqrt{1+Y^2}}{Y\sqrt{1+X^2}}$

$$YY'\sqrt{1+X^2} - X\sqrt{1+Y^2} = 0$$

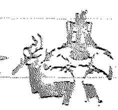
$$Y\sqrt{1+X^2}\,dY - X\sqrt{1+Y^2}\,dx = 0$$

$$\frac{Y}{\sqrt{1+Y^2}}\,dY - \frac{X}{\sqrt{1+X^2}}\,dx = 0 \Leftarrow \text{SEPER.}$$

$$\sqrt{Y^2+1} = \sqrt{1+X^2} + C$$

$$Y^2 = \left[\sqrt{1+X^2} + C\right]^2 - 1$$

$$Y = \pm\left[\left\{\sqrt{X^2+1} + C\right\}^2 - 1\right]^{1/2} \Leftarrow \text{ANSWE}$$



1-3. $Y' = \dfrac{a^2}{(X+Y)^2}$

$$(X+Y)^2\,dY - a^2\,dx = 0$$

LET $V = X + Y \Rightarrow dV = dx + dY \Leftarrow$ VAR. SUB.

$$\Rightarrow V^2(dV - dx) - a^2\,dx = 0$$

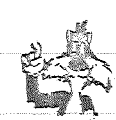
$$V^2\,dV - (V^2+a^2)\,dX = 0$$

$$\frac{V^2}{V^2+a^2}\,dV - dX = 0 \quad \Leftarrow \text{SEPERABLE}$$

$$V - a\tan^{-1}\frac{V}{a} - X = C$$

$$(X+Y) - a\tan^{-1}\left(\frac{X+Y}{a}\right) - \cancel{X} = C$$

$$Y = C + a\tan^{-1}\left(\frac{X+Y}{a}\right)$$



1-4. $Y' + Y\cos x = \frac{1}{2}\sin 2x = \sin x \cos x$

OF THE FORM

$$Y' + Y f(x) = g(x)$$

INTEGRATING FACTOR: $\lambda(x) = e^{\int f(x)dx}$

$$= e^{\int \cos x\, dx}$$

$$= e^{\sin x}$$

$$dY + Y\cos x\, dx = \sin x \cos x\, dx$$

EXACT $\Rightarrow e^{\sin x} dY + Y\cos x\, e^{\sin x} dx = \sin x \cos x\, e^{\sin x} dx$

LET $I = \int \sin x \cos x\, e^{\sin x} dx$

$$U = \sin x \qquad dV = \cos x\, e^{\sin x} dx$$

$$dU = \cos x\, dx \qquad V = e^{\sin x}$$

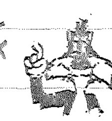
$\Rightarrow I = \sin x\, e^{\sin x} - \int \cos x\, e^{\sin x} dx$

$= \sin x\, e^{\sin x} - e^{\sin x} + C$

$= [\sin x - 1]\, e^{\sin x} + C$

THUS:

$Y e^{\sin x} = [\sin x - 1]\, e^{\sin x} + C$

$\Rightarrow Y = [\sin x - 1] + Ce^{-\sin x}$ 

1-5. $(1-x^2)Y' - xY = xY^2$

$$(1-x^2)\,dY - (xY + xY^2) = 0$$

$$(1-x^2)\,dY - xY(1+Y) = 0$$

$$\frac{1}{Y(Y+1)}\,dY - \frac{x}{1-x^2}\,dx = 0 \Leftarrow \text{SEPER.}$$

$$-\ln\frac{Y+1}{Y} + \frac{1}{2}\ln(1-x^2) = C_1$$

$$\ln\frac{Y}{Y+1} + \ln(1-x^2)^{\frac{1}{2}} = C_1$$

$$\ln\left[\frac{Y\sqrt{1-x^2}}{Y+1}\right] = C_1$$

$$\Rightarrow \frac{Y\sqrt{1-x^2}}{Y+1} = e^{C_1} = C_2$$

$$\frac{Y+1}{Y\sqrt{1-x^2}} = \frac{1}{C_2} = C$$

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{Y\sqrt{1-x^2}} = C$$

$$\frac{1}{Y\sqrt{1-x^2}} = C - \frac{1}{\sqrt{1-x^2}}$$

$$Y\sqrt{1-x^2} = \frac{1}{C - 1/\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}}{C\sqrt{1-x^2} - 1}$$

$$\therefore\ Y = \frac{1}{C\sqrt{1-x^2} - 1}$$

1-6. $2X^3Y' = 1 + \sqrt{1+4X^2Y}$

$$2X^3 dY - \left[1 + \sqrt{1+4X^2Y}\right] dX = 0$$

MAKE ISOBARIC: LET $V = \sqrt{1+4X^2Y}$

$$\Rightarrow Y = \frac{V^2-1}{4X^2}$$

$$dY = \frac{4X^2(2V\,dV) - (V^2-1)(8X)dX}{16X^4}$$

$$= \frac{XV\,dV - (V^2-1)dX}{2X^3}$$

$$= \frac{V}{2X^2}dV - \frac{V^2-1}{2X^3}dX$$

$$\Rightarrow 2X^3\left[\frac{V}{2X^2}dV - \frac{V^2-1}{2X^3}dX\right] - [1+V]dX = 0$$

$$XV\,dV - [(V^2-1) + (V+1)]dX = 0$$

$$XV\,dV - [V^2+V]dX = 0$$

$$XV\,dV - V(V+1)dX = 0$$

$$X\,dV - (V+1)dX = 0$$

$$\frac{dV}{V+1} - \frac{1}{X}dX = 0$$

$$\ln(V+1) - \ln X = C_1$$

$$\ln\left[\sqrt{1+4X^2Y} + 1\right] - \ln X = C_1$$

$$\ln\left[\frac{\sqrt{1+4X^2Y}+1}{X}\right] = C_1$$

$$\sqrt{1+4X^2Y} = Xe^{C_1} - 1 = CX - 1$$

$$4X^2Y = (CX-1)^2 - 1$$

$$\Rightarrow Y = \frac{(CX-1)^2-1}{4X^2}$$

$$= \frac{C^2X^2 - 2CX}{4X^2} = \frac{C^2X - 2C}{4X}$$

$$= \left(\frac{C}{2}\right)^2 - \frac{C}{2X}$$

$$= C_2^2 - C_2/X$$

$$\Rightarrow Y = C_2\left(C_2 - 1/X\right)$$

1-7. $Y'' + Y'^2 + 1 = 0$

LET $V = Y'$

$\Rightarrow V' + V^2 + 1 = 0$

$dV + (V^2 + 1)\,dx = 0$

$\frac{dV}{V^2+1} + dx = 0$

$\tan^{-1} V + x = C$

$\tan^{-1} V = C - x$

$V = \tan(C - x)$

$= -\tan(x - C)$

$Y = \int V\,dx = -\int \tan(x - C)$

$= -\left[-\ln\cos(x - C)\right]$

ANS. $\Rightarrow Y = \ln\cos(x - C)$

1-8. $Y'' = e^Y$

$$Y'Y'' = Y'e^Y \Rightarrow \int Y'Y''dx = \int Y'e^Y dx$$

$$\Rightarrow (Y')^2 = 2e^Y + c^2 \Rightarrow Y' = \pm[2e^Y + c^2]^{1/2}$$

$$\therefore \frac{dY}{\pm[2e^Y + c^2]^{1/2}} - dx = 0$$

LET $I = \int \frac{dY}{[2e^Y + c^2]^{1/2}}$

$$V^2 = 2e^Y + c^2 \Rightarrow e^Y = \frac{V^2 - c^2}{2} \Rightarrow Y = \ln \frac{V^2 - c^2}{2}$$

$$\frac{dY}{dV} = \frac{2}{V^2 - c^2}\frac{d}{dV}\frac{V^2}{2} \Rightarrow dY = \frac{2V\,dV}{V^2 - c^2}$$

THEN $I = \int \left(\frac{1}{V}\right)\left(\frac{2V}{V^2 - c^2}\right)dV = 2\int \frac{dV}{V^2 - c^2}$

$$= 2\left[\frac{1}{2c}\ln\frac{V-c}{V+c}\right] = \frac{1}{c}\ln\left(\frac{V-c}{V+c}\right)$$

$$\Rightarrow \pm\frac{1}{c}\ln\frac{V-c}{V+c} = x + k \Rightarrow \ln\frac{V-c}{V+c} = \pm c(x+k)$$

$$\frac{V-c}{V+c} = e^{\pm c(x+k)} \Rightarrow V - c = (V+c)e^{\pm c(x+k)}$$

$$V\left[1 - e^{\pm c(x+k)}\right] = c\left[1 + e^{\pm c(x+k)}\right]$$

$$V = c\left[\frac{1 + e^{\pm c(x+k)}}{1 - e^{\pm c(x+k)}}\right] = c\left[\frac{e^{\mp c(x+k)/2} + e^{\pm c(x+k)/2}}{e^{\mp c(x+k)/2} - e^{\pm c(x+k)/2}}\right]$$

$$= c\left[\frac{e^{\pm c(x+k)/2} + e^{\mp c(x+k)/2}}{e^{\pm c(x+k)/2} - e^{\mp c(x+k)/2}}\right] = c\coth\left(\frac{\pm c(x+k)}{2}\right)$$

$$= \pm c\coth\left[\frac{c(x+k)}{2}\right] = \pm[2e^Y + c^2]^{1/2}$$

THEN $c^2\coth^2\left[\frac{c(x+k)}{2}\right] - c^2 = 2e^Y$

$$\Rightarrow e^Y = \frac{c^2}{2}\left[\coth^2\left[\frac{c(x+k)}{2}\right] - 1\right] = \frac{c^2}{2}\,\mathrm{csch}^2\left[\frac{c(x+k)}{2}\right]$$

$$\Rightarrow Y = \ln \frac{c^2}{2}\,\mathrm{csch}^2\left[\frac{c(x+k)}{2}\right]$$

$$= 2\ln\frac{c}{\sqrt{2}}\,\mathrm{csch}\left[\frac{c(x+k)}{2}\right]$$

1-9. $x(1-x)Y'' + 4Y' + 2Y = 0$

$$Y'' + \frac{4}{x(1-x)}Y' + \frac{2}{x(1-x)}Y = 0$$

LET $f(x) = \frac{4}{x(1-x)}$ $\quad g(x) = \frac{2}{x(1-x)}$

AND $Y = V(x)P(x) \ni P(x) = e^{-\frac{1}{2}\int f(x)dx}$

$$-\frac{1}{2}\int f(x)dx = 2\int \frac{dx}{x(x-1)} = 2\ln\frac{x-1}{x}$$

$$\Rightarrow P = \left(\frac{x-1}{x}\right)^2$$

$$P' = \frac{2(x-1)}{x^3}$$

$$P'' = 2\,\frac{x^2-3x+3}{x^4}$$

$$V'' + \left[2\frac{P'}{P} + f\right]V' + \left[\frac{P'' + fP' + gP}{P}\right]V = 0$$

$$V'' + \left[2\,\frac{2(x-1)\cancel{x^2}}{x^3(x-1)^2} - \frac{4}{x(x-1)}\right]V'$$
$$+ \frac{x^2}{(x-1)^2}\left[2\,\frac{x^2-3x+3}{x^4} - \frac{4\cdot 2\cancel{(1-x)}}{x\cancel{(1-x)}x^3} + \frac{2(1-x)^{\cancel{2}}}{x\cancel{(1-x)}x^2}\right]V = 0$$

$$V'' + \frac{x^2}{(x-1)^2}\left[2\,\frac{x^2-3x+3}{x^4} - \frac{8}{x^4} + \frac{2(1-x)}{x^3}\right]V = 0$$

$$V'' + \frac{1}{x^2(x-1)^2}\left[(2x^2-6x+6) - 8 + (2x-2x^2)\right]V = 0$$

$$V'' + \frac{1}{x^2(x-1)^2}\left[-4x-2\right] = 0$$

$$V'' - \frac{2(2x+1)}{x^2(x-1)^2}V = 0$$

$$\Rightarrow \frac{d^2V}{V} = \frac{2(2x+1)}{x^2(x-1)^2}d\,x^2 = J(x)$$

(CONT →)

LET $I = \int \frac{2(2X+1)dx}{X^2(X-1)^2}$ AND $J = \int I\,dX$

$\Rightarrow I = 2\left[\int \frac{2dx}{X(X-1)^2} + \int \frac{dx}{X^2(X-1)^2}\right]$

$= 2[I_1 + I_2]$

$I_1 = \int \frac{2dx}{X(X-1)^2} = \frac{-2}{X-1} + 2\ln\frac{X}{X-1}$

$I_2 = \int \frac{dx}{X^2(X-1)^2} = \frac{-1}{X-1} - \frac{1}{X} - 2\ln\frac{X-1}{X}$

$\Rightarrow I = \left(\frac{-4}{X-1} + 4\ln\frac{X}{X-1}\right) - \left(\frac{2}{X-1} + \frac{2}{X} + 4\ln\frac{X-1}{X}\right) + C$

$= \frac{-6}{X-1} - \frac{2}{X} + 4\ln\frac{X}{X-1} + 4\ln\frac{X}{X-1} + C$

$= \frac{-6}{X-1} - \frac{2}{X} + 8\ln\frac{X}{X-1} + C$

$\Rightarrow J = \int I\,dX = -6\ln(X-1) - 2\ln X + CX + I_3 + C_1$

$I_3 = 8\int \ln\frac{X}{X-1}\,dX$

$= 8\int \ln X\,dX - 8\int \ln(X-1)\,dX$

$= I_4 - I_5$

$I_4 = 8\int \ln X\,dX = 8X\ln X - 8X$

$I_5 = 8\int \ln(X-1)\,dX = 8(X-1)\ln(X-1) - 8X$

$\Rightarrow I_3 = 8X\ln X - 8(X-1)\ln(X-1)$

$\therefore J = -6\ln(X-1) - 2\ln X + 8X\ln X - 8(X-1)\ln(X-1) + CX + C_1$

$= [-6 - 8X + 8]\ln(X-1) + [8X - 2]\ln X + CX + C_1$

$= [2 - 8X]\ln(X-1) - (2-8X)\ln X + CX + C_1$

$= (2-8X)[\ln(X-1) - \ln X] + CX + C_1$

$= 2(1-4X)\ln\frac{X-1}{X} + CX + C_1$

NOW $\int\int \frac{d^2V}{V} = \int\int \left[\frac{dV}{V}\right] dV = \int \ln V\,dV$

$= V\ln V - V$ (CONT.)

THUS, TO SUMMARIZE TO DATE:

$$Y = PV$$
$$= \left(\frac{X-1}{X}\right)^2 Y$$

WHERE

$$V\left[\ln V - 1\right] = 2(4X-1)\ln\left(\frac{X}{X-1}\right) + CX + C_1$$

ONWARD:

$$V = \left(\frac{X}{X-1}\right)^2 Y$$
$$\Rightarrow \ln V = 2\ln\left(\frac{X}{X-1}\right) + \ln Y$$

THEN

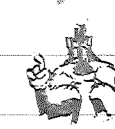
$$\left(\frac{X}{X-1}\right)^2 Y\left[2\ln\left(\frac{X}{X-1}\right) + \ln Y - 1\right] = 2(4X-1)\ln\frac{X}{X-1} + CX + C_1$$

OR

$$2\left[\left(\frac{X}{X-1}\right)^2 Y - (4X-1)\right]\ln\frac{X}{X-1} = \left[1 - \ln Y\right]\left(\frac{X}{X-1}\right)^2 Y + CX + C_1$$

OR

$$2\left[\left(\frac{X}{X-1}\right)^2 Y - 4X + 1\right]\ln\left(\frac{X}{X-1}\right) = \left(1 - \ln Y\right)\left(\frac{X}{X-1}\right)^2 Y + CX + C_1$$



Q.E.D.

1-10. $(1-X)Y^2dX - X^3dY = 0$

$\left(\frac{1-X}{X^3}\right)dX - \frac{1}{Y^2}dY = 0$ ⇐ SEPERABLE

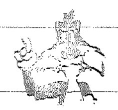
$\left(\frac{1}{X^3} - \frac{1}{X^2}\right)dX - \frac{1}{Y^2}dY = 0$

$\frac{-1}{2X^2} + \frac{1}{X} + \frac{1}{Y} = C$

$\frac{1}{Y} = C_1 + \frac{1}{2X^2} - \frac{1}{X}$

$= \frac{2C_1X^2+1-2X}{2X^2}$

$\Rightarrow Y = \frac{2X^2}{CX^2-2X+1}$



1-11. $xy' + y + x^4 y^4 e^x = 0$

$y' + \frac{1}{x} y = -x^4 e^x y^4 \Leftarrow$ BERNOULLI EQ

$\frac{y'}{y^4} + \frac{1}{x y^3} = -x^4 e^x$

$v = \frac{1}{y^3} \Rightarrow dv = \frac{-3\,dy}{y^4} \Rightarrow dy = -\frac{1}{3} y^4 dv$

$\frac{dy}{y^4} + \frac{dx}{x y^3} = -x^4 e^x dx$

$-\frac{1}{3} dv + \frac{v\,dx}{x} = -x^4 e^x dx$

$\frac{dv}{dx} - \frac{3v}{x} = 3x^4 e^x$

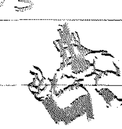
$\lambda(x) = e^{\int f(x)dx} = e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$

$dv - \frac{3v}{x} dx = 3x^4 e^x dx$

$\frac{dv}{x^3} - \frac{3v}{x^4} dx = 3x e^x dx \Leftarrow$ EXACT!

$\frac{v}{x^3} = \int 3x e^x dx = 3e^x(x-1) + C$

$\Rightarrow v = [3e^x(x-1) + C]\, x^3 = \frac{1}{y^3}$

$\Rightarrow y = \frac{1}{x} [3e^x(x-1) + C]^{-1/3}$ 

1-12. $(1+X^2)Y' + Y = \operatorname{Tan}^{-1} X$

LET $\theta = \tan^{-1} X \Rightarrow X = \tan\theta$

$(1+\tan^2\theta)\,dY + Y\,dX = \theta\,dX$

$dX = \sec^2\theta\, d\theta$

$\Rightarrow \sec^2\theta\, dY + Y\sec^2\theta\, d\theta = \theta\sec^2\theta\, d\theta$

$\Rightarrow dY + Y\,d\theta = \theta\, d\theta$

$Y' + Y = \theta$

COMPLEMENTARY SOLUTION IS $Y = Ce^{-\theta}$

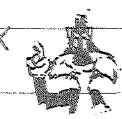
TO THE PARTICULAR SOLUTION

$\lambda(\theta) = e^{\theta}$

$e^{\theta} dY + Ye^{\theta} d\theta = \theta e^{\theta} d\theta$

$\Rightarrow Ye^{\theta} = e^{\theta}(\theta - 1) + C_2$

$\Rightarrow Y = (\theta - 1) + C_2 e^{-\theta}$

$= \operatorname{Tan}^{-1} X - 1 + C_2 e^{\tan^{-1} X}$ 

1-13. $X^2Y'^2 - 2(XY-4)Y' + Y^2 = 0$

$X^2Y'^2 - 2XYY' + 8Y' + Y^2 = 0$

$(\cancel{2XY'^2} + 2X^2Y'Y'') - (2YY' + \cancel{2XY'^2} + 2XYY'') + 8Y'' + \cancel{2YY'} = 0$

$(2X^2Y' - 2XY + 8)Y'' = 0 \Leftarrow$ CLAIRAUT-TYPE EQN.

SINGULAR SOLUTION:

$2X^2Y' - 2XY = -8$

$2X^2dY + (8 - 2XY)dX = 0 \leftarrow$ ISOBARIC

$m+2 = 1 = m+2 \Rightarrow m = -1$

$\Rightarrow$ LET $Y = V/X \Rightarrow dY = \frac{XdV - VdX}{X^2} = \frac{dV}{X} - \frac{V}{X^2}dX$

$2X^2\left(\frac{dV}{X} - \frac{VdX}{X^2}\right) + (8-2V)dX = 0$

$2XdV - 2VdX + (8-2V)dX = 0$

$2XdV + [8-4V]dX = 0 \Rightarrow XdV + 2[2-V]dX = 0$

$\frac{dV}{2(2-V)} + \frac{dX}{X} = 0 \Leftarrow$ SEPERATED

$-\frac{1}{2}\ln(V-2) + \ln X = C_1$

$\ln\frac{X}{\sqrt{V-2}} = C_1 \Rightarrow \frac{X}{\sqrt{V-2}} = C_0$

OR $X = C_0\sqrt{XY-2} \Rightarrow X^2 = C'(XY-2)$

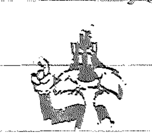
$C'XY = X^2 + 2C' \Rightarrow Y = \frac{X^2 + 2C'}{C'X} = \frac{X}{C'} + \frac{2}{X} = CX + \frac{2}{X} \leftarrow$ SINGULAR

GENERAL SOLUTION IS $Y = aX + b$

PLUGGING INTO INITIAL D.E. GIVES $a = \frac{-b^2}{8}$

$\Rightarrow Y = \frac{-b^2}{8}X + b \leftarrow$ GENERAL

$\therefore$ IF YA WANNA PUT IT TOGETHER: $Y = C'X + b + \frac{2}{X}$



BOB MARKS

1.14. $YY'' - Y'^2 - 6XY^2 = 0$

$$\frac{Y''}{Y} - \frac{Y'^2}{Y^2} = 6X$$

$$\frac{YY'' - Y'^2}{Y^2} = 6X$$

$$\frac{d}{dx}\left(\frac{Y'}{Y}\right) = \frac{d}{dx} 3X^2 + C$$

$$\frac{Y'}{Y} = 3X^2 + C$$

$$Y' - Y(3X^2 + C) = 0$$

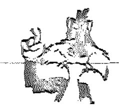
$$\frac{dY}{Y} - (3X^2 + C)dx = 0 \Leftarrow \text{SEPERATED}$$

$$\ln Y - (X^3 + CX) = C_2$$

$$\ln Y = X^3 + CX + C_2$$

$$Y = e^{X^3 + CX + C_2}$$

$$= C_3 e^{X^3 + CX}$$



1.15. $X^4 YY'' + X^4 Y'^2 + 3X^3 YY' - 1 = 0$

LET $V = X^4 YY' \Rightarrow \frac{V}{X} = X^3 YY'$

$V' = 4X^3 YY' + X^4 YY'' + X^4 Y'^2$

$\Rightarrow V' - X^3 YY' - 1 = 0$

$V' - \frac{V}{X} - 1 = 0$

$XV' - (V + X) = 0 \Rightarrow X\,dV - (V + X)\,dX = 0$

INTEGRATING FACTOR: $\lambda = \frac{1}{X^2}$

$\Rightarrow \frac{1}{X} dV - \left(\frac{V}{X^2} + \frac{1}{X}\right) dX = 0 \Leftarrow$ EXACT

$\Rightarrow \frac{-V}{X} + \ln X = C_0$

$V = -X \ln C_1 X = X^4 YY'$

$YY' = -\frac{1}{X^3} \ln C_1 X$

$Y\,dY + \frac{1}{X^3} \ln C_1 X\, dX = 0 \Leftarrow$ SEPERATED

$Y^2 = -2 \int \frac{1}{X^3} \ln C_1 X\, dX + C_2$

$= \frac{\ln CX}{X^2} + \frac{1}{2X^2} + C_2$

1.16. $X^2Y'' - 2Y = X$ (STAIGHT FROM BOOK, Pg 10)

COMP. SOL. FROM

$$X^2Y'' - 2Y = 0$$

$$\Rightarrow Y = C_1X^2 + \frac{C_2}{X}$$

LET $Y = U_1X^2 + \frac{U_2}{X}$

$$Y' = 2XU_1 - \frac{1}{X^2}U_2 + X^2U_1' + \frac{1}{X}U_2'$$

LET $X^2U_1' + \frac{1}{X}U_2' = 0$

$$\Rightarrow Y' = 2XU_1 - \frac{1}{X^2}U_2$$

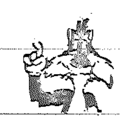
$$Y'' = 2U_1 + \frac{2}{X^3}U_2 + 2XU_1' - \frac{1}{X^2}U'$$

GIVES $2X^3U_1' - U_2' = X$

$$U_1' = \frac{1}{3X^2} \quad ; \quad U_2' = -\frac{X}{3}$$

$$U_1 = -\frac{1}{3X} + C_1 \quad ; \quad U_2 = -\frac{X^2}{6} + C_2$$

$$\Rightarrow Y = -\frac{X}{2} + C_1X^2 + \frac{C_2}{X}$$



1.17. $y''' - 2y'' - y' + 2y = \sin x$

COMPLEMENTARY SOLUTION FROM

$$y''' - 2y'' - y' + 2y = 0$$

$$m^3 - 2m^2 - m + 2 = 0$$

$$(m^2 - 1)(m - 2) = 0$$

$$(m-1)(m+1)(m-2) = 0$$

$$\Rightarrow m = 1, -1, 2$$

$$y_c = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}$$

THE PARTICULAR SOLUTION IS

(BY METHOD OF UNDETERMINED COEFFICIENTS

$$y_p = \frac{1}{10}\cos x + \frac{1}{5}\sin x$$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + \frac{1}{5}\sin x + \frac{1}{10}\cos x$$

1-18. $Y^{iv} + 2Y'' + Y = \cos X$

$[D^4 + 2D^2 + 1]\, Y = \cos X$

COMP. SOLUTION:

$[D^4 + 2D^2 + 1]\, Y = 0$

$[D^2 + 1]^2\, Y = 0$

$[(D+i)(D-i)]^2 = 0$

$Y_c = A_1 e^{ix} + B_1 e^{-ix} + CXe^{ix} + DXe^{-ix}$

OR EQUIVALENTLY:

$Y_c = A \cos X + B \sin X + CX \cos X + DX \sin X$

FOR PARTICULAR SOLUTION, USE LAPLACE

$Y^{iv} + 2Y'' + Y = \cos X$

$[S^4 + 2S^2 + 1]\, \bar{Y}(S) = \frac{S}{S^2 + a^2}$

$(S^2 + 1)^2\, \bar{Y}(S) = \frac{S}{S^2 + a^2}$

$\bar{Y}(S) = \frac{S}{(S^2 + a^2)^3}$

$\Rightarrow Y(X) = \frac{1}{8}\left[X \sin X - X^2 \cos X\right]$

SO THE WHOLE THING IS

$Y = A \cos X + \bar{B} \sin X + CX \cos X + DX \sin X$

$+ \frac{X}{8} \sin X - \frac{X^2}{8} \cos X$

1.19 $Y'' + 3Y' + 2Y = e^{e^x}$

COMP. SOL: $Y'' + 3Y' + 2Y = 0$

$m^2 + 3m + 2 = (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$Y_c = C_1 e^{-x} + C_2 e^{-2x}$

NOW: $(D^2 + 3D + 2)Y = e^{e^x}$

$(D+1)(D+2)Y = e^{e^x}$

$(D+2)Y = \dfrac{e^{e^x}}{D+1}$

NOW $(D+1)Y_1 = e^{e^x}$

$Y_1' + Y_1 = e^{e^x}$

$dY_1 + Y_1 dx = e^{e^x}$

$e^x dY_1 + e^x Y_1 dx = e^x e^{e^x} \Leftarrow$ XACT

$e^x Y_1 = \int e^x e^{e^x} dx = e^{e^x} \Rightarrow Y_1 = e^{-x} e^{e^x}$

$\Rightarrow (D+2)Y = e^{-x} e^{e^x}$

$Y' + 2Y = e^{-x} e^{e^x}$

$dY + 2Y dx = e^{-x} e^{e^x} dx$

$e^{2x} dY + 2Y e^{2x} dx = e^x e^{e^x} dx \Leftarrow$ XACT

$Y e^{2x} = \int e^x e^{e^x} dx = e^{e^x}$

$\Rightarrow Y = e^{-2x} e^{e^x}$

SO TOTAL SOLUTION IS

$Y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} e^{e^x}$

1-20. $a^2 y''^2 = (1 + y'^2)^3$

$V = Y'$

$\Rightarrow a^2 v'^2 = (1+v)^3$

$av' = \pm (1+v)^{3/2}$

$av' \pm (1+v)^{3/2} = 0$

$a\,dv \pm (1+v)^{3/2} dx = 0$

$\pm a (1+v)^{-3/2} dv + dx = 0 \Leftarrow$ SEPERATED

$\int \frac{dv}{(1+v)^{3/2}} = -2(1+v)^{-1/2}$

$\Rightarrow \pm \frac{2}{\sqrt{v+1}} \mp x = c$

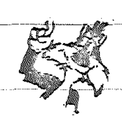
$\pm \frac{2}{\sqrt{v+1}} = c - x$

$\frac{4}{v+1} = (x-c)^2$

$\frac{v+1}{4} = \frac{1}{(x-c)^2}$

$V = Y' = \frac{4}{(x-c)^2} - 1$

$Y = \frac{-4}{x-c} - x + C_2$



1.21. $$R\frac{dq}{dt} + \frac{1}{C}q = V_0\left(\frac{t}{\tau}\right)^2 e^{-t/\tau}$$

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{V_0}{R\tau^2} t^2 e^{-t/\tau}$$

$$\lambda(t) = e^{+t/RC} \Leftarrow \text{INT. FACTOR}$$

XACT! $$\Rightarrow e^{t/RC}dq + \frac{1}{RC}q e^{t/RC}dt = \frac{V_0}{R\tau^2} t^2 e^{t\left[\frac{1}{RC}-\frac{1}{\tau}\right]} dt$$

$$\Rightarrow q e^{t/RC} = \frac{V_0}{R\tau^2}\int t^2 e^{t\left(\frac{1}{RC}-\frac{1}{\tau}\right)} dt$$

$$I = \int t^2 e^{at} dt \quad ; \; a = \frac{1}{RC} - \frac{1}{\tau} = \frac{\tau - RC}{RC\tau}$$

$$u = t^2 \qquad dv = e^{at}dt$$

$$du = 2t\,dt \qquad v = \frac{1}{a}e^{at}$$

$$I = \frac{t^2}{a}e^{at} - \int \frac{2t}{a}e^{at}dt$$

$$u = \frac{2t}{a} \qquad dv = e^{at}dt$$

$$du = \frac{2}{a} \qquad v = \frac{1}{a}e^{at}$$

$$I = \frac{t^2}{a}e^{at} - \frac{2t}{a^2}e^{at} + \int \frac{2}{a^2}e^{at}$$

$$= \frac{t^2}{a}e^{at} - \frac{2t}{a^2}e^{at} + \frac{2}{a^3}e^{at} + C$$

$$= \left[\frac{t^2}{a} - \frac{2t}{a^2} + \frac{2}{a^3}\right]e^{at} + C$$

$$= \left[\frac{RC\tau}{\tau - RC}t^2 - 2\left(\frac{RC\tau}{\tau-RC}\right)^2 t + 2\left(\frac{RC\tau}{\tau-RC}\right)^3\right]e^{t\left[\frac{1}{RC}-\frac{1}{\tau}\right]} + C$$

$$\Rightarrow q e^{t/RC} = \frac{V_0}{R\tau^2}\left[\frac{RC\tau}{\tau-RC}t^2 - 2\left(\frac{RC\tau}{\tau-RC}\right)^2 t + 2\left(\frac{RC\tau}{\tau-RC}\right)^3\right]e^{t\left[\frac{1}{RC}-\frac{1}{\tau}\right]} + C_1$$

$$\therefore q = \frac{V_0}{R\tau^2}\left[\frac{RC\tau}{\tau-RC}t^2 - 2\left(\frac{RC\tau}{\tau-RC}\right)^2 t + 2\left(\frac{RC\tau}{\tau-RC}\right)^3\right]e^{-t/\tau} + Ce^{-\frac{t}{RC}}$$

$$q(0) = 0 = \frac{2V_0}{R\tau^2}\left(\frac{RC\tau}{\tau-RC}\right)^3 + C_1$$

$$= \frac{2V_0 R^3C^3\tau^3}{R\tau^2(\tau-RC)^3} + C_1$$

$$= \frac{2V_0R^2C^3\tau}{(\tau-RC)^3} + C_1 \Rightarrow C_1 = \frac{-2V_0R^2C^3\tau}{(\tau-RC)^3}$$

AND

$$q(t) = \frac{V_0}{R\tau^2}\left[\left(\frac{RC\tau}{\tau-RC}\right)t^2 - 2\left(\frac{RC\tau}{\tau-RC}\right)^2 t + 2\left(\frac{RC\tau}{\tau-RC}\right)^3\right]e^{-t/\tau} - \frac{2V_0R^2C^3\tau}{(\tau-RC)^3}e^{-t/RC}$$

1-22. $\frac{dN}{dt} = \lambda(N_s - N)$

$dN + \lambda(N - N_s)dt = 0$

$\frac{dN}{N - N_s} + \lambda dt = 0 \Leftarrow$ SEPERATED

$\ln N - N_s + \lambda t = C_1$

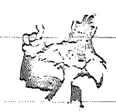
$\ln N - N_s = C_1 - \lambda t$

$N - N_s = Ce^{-\lambda t}$

$N(t) = N_s + Ce^{-\lambda t}$

$N(0) = 0 = N_s + C \Rightarrow C = -N_s$

$\Rightarrow N(t) = N_s(1 - e^{-\lambda t})$



1.23. $AY'' + A'Y' + \frac{Y}{A} = 0$

LET $Y = e^{v}$

$Y' = v'e^{v}$

$Y'' = (v'^2 + v'')e^{v}$

$\Rightarrow A(v'^2 + v'') + A'v' + \frac{1}{A} = 0$

LET $\varphi = v'$

$\Rightarrow A(\varphi^2 + \varphi') + A'\varphi + \frac{1}{A} = 0$

$A\varphi' + A'\varphi + A\varphi^2 + \frac{1}{A} = 0$

LET $\psi = A\varphi \Rightarrow \varphi = \frac{\psi}{A}$

$\psi' = A'\varphi + \varphi'A$

$\psi' + \frac{\psi^2}{A} + \frac{1}{A} = 0$

$\psi' + \frac{1}{A}(\psi^2 + 1) = 0$

$\frac{d\psi}{\psi^2 + 1} + \frac{dx}{A} = 0$

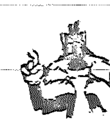
$\tan^{-1}\psi + \int \frac{dx}{A} = C_1$

$\psi = A\varphi = \tan\left[C_1 + \int \frac{dx}{A}\right]$

$\varphi = V' = \frac{1}{A}\tan\left[C + \int \frac{dx}{a}\right]$

$\ln Y = V = \int \frac{1}{A}\tan\left[C + \int \frac{dx}{A}\right]dx + C_3$

$\Rightarrow Y = C_2 e^{\int \frac{1}{A}\tan\left[C_1 + \int \frac{dx}{A}\right]dx}$



1-24. $xY'' + 2Y' + n^2 x Y = \sin \omega x$

$Y'' + \frac{2}{x} Y' + n^2 Y = \frac{\sin \omega x}{x}$

LET $Y = VP \Rightarrow P = e^{-\frac{1}{2}\int f(x)dx}$

$P = e^{-\frac{1}{2}\int \frac{2}{x} dx} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

$P = \frac{1}{x}, \quad P' = -\frac{1}{x^2}, \quad P'' = \frac{2}{x^3}$

GIVES: $V'' + n^2 V = \frac{\sin \omega x}{Px} = \sin \omega x$

COMP. SOLUTION: $V_c = A \sin nx + B \cos nx$

$V'' + n^2 V = \sin \omega x$

$\xi = \omega x \Rightarrow \frac{d^2V}{d\xi^2} = \omega^2 \frac{d^2V}{dx^2}$

$\omega^2 V'' + n^2 V = \sin \xi$

USE METHOD OF UNDETERMINED COEFF.:

$V = C_1 \sin \xi + C_2 \cos \xi, \quad V'' = -V$

$(n^2 - \omega^2)(C_1 \sin \xi + C_2 \cos \xi) = \sin \xi$

$C_2 = 0, \quad C_1 = \frac{1}{n^2 - \omega^2}$

$\Rightarrow V = \frac{1}{n^2 - \omega^2} \sin \omega x$

SO THE WHOLE SOLUTION IS

$Y = \frac{A}{x} \sin nx + \frac{B}{x} \cos nx + \frac{1}{x} \frac{1}{(n^2 - \omega^2)} \sin \omega x$

1-25. $(1-x)y'' + xy' - y = (1-x)^2$

$y_c = x$

LET $y = xp$

$y' = p + xp'$

$y'' = 2p' + xp''$

$(1-x)(2p' + xp'') + x(p + xp') - xp = (1-x)^2$

$2p' + xp'' - 2xp' - x^2p'' + \cancel{xp} + x^2p' - \cancel{xp} = (1-x)^2$

$(x - x^2)p'' + (2 - 2x + x^2)p' = (1-x)^2$

LET $q = p'$

$x(1-x)q' + (x^2 - 2x + 2)q = (1-x)^2$

TRY AN INTEGRATING FACTOR $\lambda(x)$

$q' + \frac{x^2-2x+2}{x(1-x)} q = \frac{1-x}{x}$

$\lambda = e^{\int \frac{x^2-2x+2}{x(1-x)}dx} = e^{\int\left[\frac{x}{1-x} + \frac{2}{x-1} + \frac{2}{x(1-x)}\right]dx}$

$= e^{-x - \ln(1-x) + 2\ln(x-1) + 2\ln\frac{x}{1-x}}$

$= e^{-x} e^{\ln\left(\frac{1}{1-x}\right)} e^{\ln(x-1)^2} e^{\ln\left(\frac{x}{1-x}\right)^2}$

$= \frac{1}{(1-x)}(x-1)^2 \frac{x^2}{(1-x)^2} e^{-x}$

$= \frac{x^2}{1-x} e^{-x}$

$\frac{x^2}{1-x} e^{-x} dq + \frac{x^2-2x+2}{x(1-x)} \frac{x^2}{(1-x)} e^{-x} q\,dx = \frac{1-x}{x} \frac{x^2}{1-x} e^{-x}$

XACT $\Rightarrow \frac{x^2}{1-x} e^{-x} dq + \frac{x(x^2-2x+2)}{(1-x)^2} e^{-x} q\,dx = x e^{-x} dx$

$\frac{x^2}{1-x} q e^{-x} = \int x e^{-x} = -e^{-x}(x+1) + C$

$\frac{x^2}{1-x} q = -(x+1) + C_1 e^x$

$p' = q = \frac{(x+1)(x-1)}{x^2} + C_1 \frac{(1-x)}{x^2} e^x$

$p = \int \frac{x^2-1}{x^2} dx + C_1 \int \frac{(1-x)}{x^2} e^x dx$

$= \int\left(1 - \frac{1}{x^2}\right)dx + C_1 \int \frac{1-x}{x^2} e^x dx$

$= x + \frac{1}{x} + C_1 \int \frac{1-x}{x^2} e^x dx + C_2$

$y = xp = x^2 + C_2 x + 1 + C_1 x \int \frac{1-x}{x^2} e^x dx$

(CONT $\rightarrow$)

$$\int\left(\frac{1}{x^2}-\frac{1}{x}\right)e^x\,dx = \int\frac{e^x}{x^2}\,dx - \int\frac{e^x}{x}\,dx$$

$$\int\frac{e^x}{x^2}\,dx = -\frac{1}{x}e^x + \int\frac{1}{x}e^x\,dx$$

$$\Rightarrow \int\left(\frac{1}{x^2}-\frac{1}{x}\right)e^x\,dx = -\frac{1}{x}e^x$$

AND

$$Y = C_2X + X^2 + 1 - C_1e^X$$

EXTRA PROBLEM:

FIND A $P \ni \phi$ SUCH THAT

$\vec{\nabla}\phi = PF$

WHERE $F = z(x\vec{e}_x + y\vec{e}_y)$

$\Rightarrow \frac{d}{dx}\phi\,\tilde{a}_x + \frac{d}{dy}\phi\,\tilde{a}_y + \frac{d}{dz}\phi\,\vec{a}_z = Pzx\,\vec{e}_x + Pzy\,\vec{e}_y$

$\Rightarrow \frac{d}{dx}\phi = Pzx,\quad \frac{d}{dy}\phi = Pzy,\quad \frac{d}{dz}\phi = 0$

$\frac{1}{x}\frac{d}{dx}\phi = \frac{1}{y}\frac{d}{dy}\phi = Pz$

A SOLUTION IS: $\phi = C_1 e^{C_2(x^2+y^2)}$

$$\begin{cases} \frac{d}{dx}\phi = C_1 C_2 (2x)\, e^{C_2(x^2+y^2)} \\ \frac{d}{dy}\phi = C_1 C_2 (2y)\, e^{C_2(x^2+y^2)} \end{cases}$$

$\Rightarrow \frac{1}{x}\frac{d}{dx}\phi = \frac{1}{y}\frac{d}{dy}\phi = 2C_1C_2\, e^{C_2(x^2+y^2)}$

$= Pz$

$\Rightarrow P = \frac{2C_1C_2}{z} e^{C_2(x^2+y^2)}$

$\phi = C_1 e^{C_2(x^2+y^2)}$

BOB MARKS

1. $S = 1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - - + +$

$= \left(1 + \frac{1}{4}\right) - \left(\frac{1}{16} + \frac{1}{64}\right) + \left(\frac{1}{256} + \frac{1}{1024}\right) - + -$

$= \left(\frac{1}{4^0} + \frac{1}{4^1}\right) - \left(\frac{1}{4^2} + \frac{1}{4^3}\right) + \left(\frac{1}{4^4} + \frac{1}{4^5}\right) - + - +$

$= \left(\frac{5}{4}\right) - \left(\frac{5}{4^3}\right) + \left(\frac{5}{4^5}\right) - +$

$= 5 \sum_0 \left(\frac{1}{4}\right)^{2n+1} (-)^n$

$= \frac{5}{4} \sum_0 \left(\frac{1}{4}\right)^{2n} (-)^n$

$= \frac{5}{4} \sum \left(\frac{1}{16}\right)^n (-)$

$= \frac{5}{4} \frac{1}{1 + 1/16}$

$= \frac{5}{4} \cdot \frac{16}{17}$

$= \frac{20}{17} = 1.176470588$

2.2: $S = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \dots$

$$= \sum_1 \frac{1}{n(n+2)}$$

$$= \sum_1 \frac{1}{2n} - \frac{1}{2(n+2)}$$

$$= S_1 - S_2$$

$$S_1 = \frac{1}{2}\sum_1 \frac{1}{n}$$

$$S_1(x) = \frac{1}{2}\sum_1 \frac{x^n}{n}$$

$$S_1'(x) = \frac{1}{2}\sum_1 x^{n-1}$$

$$= \frac{1}{2}\sum_0 x^n$$

$$= \frac{1}{2}\cdot\frac{1}{1-x}$$

$$S_2 = \frac{1}{2}\sum_1 \frac{1}{n+2}$$

$$S_2(x) = \frac{1}{2}\sum_1 \frac{x^{n+2}}{n+2}$$

$$S_2'(x) = \frac{1}{2}\sum_1 x^{n+1}$$

$$= \frac{1}{2}\sum_0 x^{n+2}$$

$$= \frac{x^2}{2}\sum_0 x^n$$

$$= \frac{x^2}{2}\,\frac{1}{1-x}$$

$$S'(x) = S_1'(x) - S_2'(x)$$

$$2S'(x) = \frac{1}{1-x} - \frac{x^2}{1-x}$$

$$2S(x) = \int_0^x \frac{1}{1-x}\,dx - \int_0^x \frac{x^2}{1-x}$$

$$= -\ln(1-x) + \left[\tfrac{1}{2}(1-x)^2 - 2(1-x) + \ln(1-x)\right]_0^x$$

$$= -\ln(1-x) + \tfrac{1}{2}(1-x)^2 - 2(1-x) + \ln(1-x)$$

$$- \tfrac{1}{2} + 2$$

$$2S(1) = \frac{3}{2}$$

$$\Rightarrow S = 3/4$$

0
STO 1
1
STO 0
RCL 0
2
+
RCL 0
×
1/x
RCL 1
+
PAUSE
STO 1
RCL 0
1
+
STO 0
GTO 05

2-4. $\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \ldots$

$$S = \sum_0 \frac{(n+1)}{n!}$$

$$= \sum_0 \frac{n}{n!} + \sum_0 \frac{1}{n!}$$

$$= \sum_1 \frac{n}{n!} + e$$

$$= \sum_1 \frac{1}{(n-1)!} + e$$

$$= \sum_0 \frac{1}{m!} + e$$

$$= e + e$$

$$= 2e = 5.43656\,3657$$

2-5. $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots$

$= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

$= \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\right] - \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots\right]$

$= \zeta(2) - \left[\frac{1}{2^2 1^2} + \frac{1}{2^2 2^2} + \frac{1}{2^2 3^2} + \frac{1}{2^2 4^2} + \ldots\right]$

$= \zeta(2) - \frac{1}{4}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$

$= \zeta(2) - \frac{1}{4}\zeta(2)$

$= \frac{3}{4}\zeta(2)$

$= \frac{3}{4}\left(\frac{\pi^2}{6}\right)$

$= \frac{\pi^2}{8}$

$$2\text{-}6.\quad S = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \dots$$

$$= 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

$$= \left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots\right] - \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right]$$

$$= \zeta(4) - \left(\frac{1}{1 \cdot 2^4} + \frac{1}{2^4\, 2^4} + \frac{1}{3^4\, 2^4} + \dots\right]$$

$$= \zeta(4) - \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)$$

$$= \zeta(4) - \frac{1}{16}\zeta(4)$$

$$= \frac{15}{16}\zeta(4)$$

$$= \frac{15}{16}\,\frac{\pi^4}{90}$$

$$= \frac{\pi^4}{96}$$

2-7. $S = 1 - \frac{1}{4^2} + \frac{1}{9^2} - \frac{1}{16^2} + \ldots$

$= 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \ldots$

$= \left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right] - \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\right]$

$= \frac{\pi^4}{96} - \frac{1}{2^4}\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right]$ $\Leftarrow$ FROM 2-6.

$= \frac{\pi^4}{96} - \frac{1}{16}\zeta(4)$

$= \frac{\pi^4}{96} - \frac{\pi^4}{16 \cdot 90}$

$= \left(\frac{1}{96} - \frac{1}{16 \cdot 90}\right)\pi^4$

$= \left(\frac{1}{6 \cdot 16} - \frac{1}{6 \cdot 16 \cdot 15}\right)\pi^4$

$= \frac{14}{6 \cdot 16 \cdot 15}\pi^4$

$= \frac{7}{3 \cdot 16 \cdot 15}\pi^4$

$= \frac{7}{720}\pi^4$

2-8. $f(\theta) = \sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{5}\sin 3\theta + \frac{1}{7}\sin 4\theta + \dots$

$$= \sum \frac{1}{2n-1}\sin n\theta$$

$$= \sum_0 \frac{1}{2n-1}\sin n\theta$$

$$= Im \sum \frac{1}{2n-1} e^{in\theta}$$

LET $Y = e^{i\theta/2}$

$$\Rightarrow f(\theta) = Im \sum_0 \frac{Y^{2n}}{2n-1}$$

$$= Im\ Y \sum_0 \frac{Y^{2n-1}}{2n-1}$$

LET $g(Y) = \sum_0 \frac{Y^{2n-1}}{2n-1}$

$$g'(Y) = \sum_0 Y^{2n} = \frac{1}{1-Y^2}$$

$$\Rightarrow g(Y) = \int_0^Y \frac{dY}{1-Y^2}$$

$$= +\frac{1}{2}\ln\frac{1+Y}{1-Y}$$

THUS $f(\theta) = Im +\frac{Y}{2}\ln\frac{1+Y}{1-Y}$

$$= Im +\frac{1}{2}e^{i\theta/2}\ln\frac{1+e^{i\theta/2}}{1-e^{i\theta/2}}$$

$$= Im +\frac{1}{2}e^{i\theta/2}\ln\frac{e^{-i\theta/4}+e^{i\theta/4}}{e^{-i\theta/4}-e^{i\theta/4}}$$

$$= Im +\frac{1}{2}e^{i\theta/2}\ln-\left[\frac{e^{i\theta/4}+e^{-i\theta/4}}{e^{i\theta/4}-e^{-i\theta/4}}\right]$$

$$= Im +\frac{1}{2}e^{i\theta/2}\ln i\left[i\frac{e^{i\theta/4}+e^{-i\theta/4}}{e^{i\theta/4}-e^{-i\theta/4}}\right]$$

$$= Im +\frac{1}{2}e^{i\theta/2}\ln\left[i\ ctn\ \theta/4\right]$$

$$= Im +\frac{1}{2}e^{i\theta/2}\ln\left[e^{i\frac{\pi}{2}}ctn\frac{\theta}{4}\right]$$

$$= Im +\frac{1}{2}e^{i\theta/2}\left[i\frac{\pi}{2}+\ln ctn\frac{\theta}{4}\right]$$

$$+2f(\theta) = Im\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]\left[i\frac{\pi}{2}+\ln ctn\frac{\theta}{4}\right]$$

$$= \frac{\pi}{2}\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\ln ctn\frac{\theta}{4}$$

$$\Rightarrow f(\theta) = +\frac{\pi}{4}\cos\frac{\theta}{2}+\frac{1}{2}\sin\frac{\theta}{2}\ln ctn\frac{\theta}{4}$$

$$2\text{-}9.\quad S = \sum^{1} \frac{(2n+1)!!\,(2n+1)}{2^{2n+2}(2n+7)(2n+5)^2(n+1)}$$

USING RATIO TEST GIVES

$$\frac{a_{n+1}}{a_n} = \frac{(2n+3)(2n+5)^2}{2(2n+1)(2n+7)(n+2)}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \quad \text{ARG!}$$

$$2\text{-}10.\quad S = \frac{(1\cdot 3)^2}{1\cdot 1\cdot 1^2} + \frac{(1\cdot 3\cdot 5)^2}{4\cdot 2\cdot (1\cdot 2)^2} + \frac{(1\cdot 3\cdot 5\cdot 7)^2}{16\cdot 3(1\cdot 2\cdot 3)^2} + \frac{(1\cdot 3\cdot 5\cdot 7\cdot 9)^2}{64\cdot 4\cdot (1\cdot 2\cdot 3\cdot 4)^2} \ldots$$

$$S = \sum_1 \frac{[(2n+1)!!]^2}{4^{n-1}\ n\ (n!)^2}$$

$$(2n+1)!! = \frac{(2n+1)!}{2^n\, n!}$$

$$S = \sum_1 \frac{[(2n+1)!]^2}{4^{n-1}\ 2^{2n}\ (n!)^4\, n}$$

$$= \sum_1 \frac{[(2n+1)!]^2}{2^{2n-2}\ 2^{2n}\ (n!)^4}$$

$$= \sum_1 4\ \frac{[(2n+1)!]^2}{n\cdot (n!)^4}$$

THE n^{TH} TERM IS

$$a_n = 4\cdot \frac{[(2n+1)!]^2}{n\cdot (n!)^4}$$

THE $n+1^{TH}$ TERM IS

$$a_{n+1} = 4\cdot \frac{[(2n+3)!]^2}{(n+1)[(n+1)!]^4} = 4\cdot \frac{[(2n+3)(2n+2)(2n+1)!]^2}{(n+1)\,[(n+1)\,n!]^4}$$

$$= 4\cdot \frac{4(2n+3)^2(n+1)^2[(2n+1)!]^2}{(n+1)^5\,(n!)^4}$$

$$= 4\cdot \frac{4(2n+3)^2[(2n+1)!]^2}{(n+1)^3\,(n!)^4}$$

$$\frac{a_{n+1}}{a_n} = \frac{4(2n+3)^2\ \cancel{[(2n+1)!]^2}}{(n+1)^3\ \cancel{(n!)^4}} \times \frac{n\cdot \cancel{(n!)^4}}{\cancel{[(2n+1)!]^2}}$$

$$= \frac{4n(2n+3)^2}{(n+1)^3}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4n\cdot 4n^2}{n^3} = 16 > 1$$

$\Rightarrow$ SERIES DIVERGES

2-12. $f = \sec x$ $\qquad f(0) = 1$

$f' = f \tan x = f t \qquad ; t' = f^2 \qquad f'(0) = 0$

$f'' = f' t + t' f = f' t + f^3 = f t^2 + f^3 \qquad f''(0) = 1$

$f''' = 2 f t t' + f' t^2 + 3 f' f^2$

$= 2 f^3 t + f t^3 + 3 f^3 t = 5 f^3 t + f t^3 \qquad f'''(0) = 0$

$f^{IV} = 15 f' f^2 t + 5 f^3 t' + 3 f t^2 t' + f' t^3$

$= 15 f^3 t^2 + 5 f^5 + 3 f^3 t^3 + f t^4 \qquad f^{IV}(0) = 5$

$f^{V} = 45 f^2 f' t^2 + 20 f^3 t t' + 25 f' f^4$

$+ 6 f f' t^3 + 9 f^2 t' t^2 + f' t^4 + 4 f t' t^3$

$= \cancel{45 f^3 t^3} + 20 f^5 t + 25 f^5 t$

$+ 6 f^2 t^4 + 9 f^4 t^2 + f t^5 + \cancel{4 f^3 t^3}$

$= 49 f^3 t^3 + 45 f^5 t + 9 f^4 t^2 + 6 f^2 t^4 + f t^5 \qquad f^{V}(0) = 0$

$f^{VI} = 147 f^2 f' t^3 + 147 f^3 t^2 t' + 225 f' f^4 t$

$+ 45 f^5 t' + 36 f' f^3 t^2 + 18 f^4 t t' \qquad f^{VI}(0) = 45$

$+ 12 f f' t^4 + 24 f^2 t' t^3 + f' t^5 + 5 f t^4 t'$

$\sec z = 1 + \frac{z^2}{2!} + \frac{5 z^4}{4!} + \frac{45 \cdot z^6}{6!} + \cdots$

$= 1 - \frac{E_2 z^2}{2!} + \frac{E_4 z^4}{4!} - \frac{E_6 z^6}{6!}$

$(E+1)^2 + (E-1)^2 = 0 = E_2 + 2E_1 + 1 + E_2 - 2E_1 + 1 = 0$

$\Rightarrow 2E_2 = -2, \quad E_2 = -1$

$(E+1)^4 + (E-1)^4 = 0 = E_4 + 4E_3 + 6E_2 + 4E_3 + 1$

$+ E_4 - 4E_3 + 6E_2 - 4E_3 + 1$

$\Rightarrow E_4 + 6E_2 + 1 = 0$

$E_4 = 5$

$(E+1)^6 + (E-1)^6 = E_6 + \cancel{15E_5} + 20E_4 + \cancel{15E_3} + 6E_2 + \cancel{E_1} + 1$

$E_6 + 100 + 6 + 1 = 0 \Rightarrow E_6 = 95$

$(E+1)^8 \Rightarrow E_8 + 28E_6 + 70E_4 + 28E_2 + 1 = 0$

ETC

BOB MARKS
DUE 11/13/75

3-1. $\int_0^\infty \frac{e^{-ar} - e^{-br}}{r} \, dr$

CONSIDER

$$\int_0^\infty e^{-ar} \, dr = \frac{1}{a}$$

$$\int_0^\infty \left[\int_a^b e^{-ar} \, da\right] dr = \int_a^b \frac{1}{a} \, da$$

$$= \int_0^\infty + \frac{1}{r} e^{-ar} \Big|_b^a \, dr = \ln a \Big|_a^b$$

$$= \int_0^\infty \frac{e^{-ar} - e^{-br}}{r} \, dr = \ln b/a$$

3-2. $\int_0^\infty \sin bx \, dx$

$$= \lim_{a \to 0} \int_0^\infty e^{-ax} \sin bx$$

$$= \lim_{a \to 0} \mathrm{Im} \int_0^\infty e^{[-a + jb]x}$$

$$= \lim_{a \to 0} \mathrm{Im} \frac{1}{-a + jb} e^{[-a + jb]x} \Big|_0^\infty$$

$$= \lim_{a \to 0} \mathrm{Im} \frac{1}{a - jb} = \lim_{a \to 0} \mathrm{Im} \frac{+1}{a - jb}$$

$$= \lim_{a \to 0} \mathrm{Im} \frac{+a + jb}{a^2 + b^2}$$

$$= \lim_{a \to \infty} \frac{+b}{a^2 + b^2}$$

$$= 1/b$$

3-3. $I(a) = \int_0^\infty \frac{\cos ax \, dx}{x^2+1}$

$$= Re \int_0^\infty \frac{e^{-iax}}{x^2+1} \, dx$$

$$I'(a) = Re \int_0^\infty \frac{(-ix)\, e^{-iax}}{x^2+1} \, dx$$

$$I''(a) = Re \int_0^\infty \frac{-x^2 e^{-iax}}{x^2+1}$$

$$I(a) - I''(a) = Re \int_0^\infty e^{-iax} \, dx = Re \left(\frac{1}{ia} \right) = 0$$

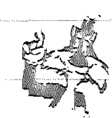
$$\Rightarrow I(a) = C e^{-a}$$

NOW $I(0) = \int_0^\infty \frac{1}{x^2+1} \, dx$

$$= \tan^{-1} x \Big|_0^\infty$$

$$= \frac{\pi}{2} \Longrightarrow C = \frac{\pi}{2}$$

$$\Rightarrow I(a) = \int_0^\infty \frac{\cos ax \, dx}{x^2+1} = \frac{\pi}{2} e^{-a}$$



3-4. $I(a) = \int_0^\infty \frac{\cos ax \, dx}{(1+x^2)^2}$

$I'(a) = \int_0^\infty \frac{-x \sin ax \, dx}{(1+x^2)^2}$

$I''(a) = \int_0^\infty \frac{-x^2 \cos ax \, dx}{(1+x^2)^2}$

$I(a) - I''(a) = \int_0^\infty \frac{\cos ax}{(1+x^2)} dx = \frac{\pi}{2} e^{-a} \Leftarrow$ FROM #3-3

INTO MORE FAMILIAR NOTATION

$Y - Y'' = \frac{\pi}{2} e^{-x}$

$Y_c = Ae^{x} + Be^{-x}$

$(D^2 - 1)Y = -\frac{\pi}{2} e^{-x}$

$(D+1)(D-1)Y = -\frac{\pi}{2} e^{-x}$

$(D+1)Y = -\frac{\pi}{2} \frac{e^{-x}}{D-1}$

$\xi' - \xi = e^{-x}$

$d\xi - \xi dx = e^{-x} dx$

$e^{-x} d\xi - \xi e^{-x} dx = e^{-2x} dx \Leftarrow$ EXACT!

$\xi e^{-x} = -\frac{1}{2} e^{-2x}$

$\xi = -\frac{1}{2} e^{-x}$

$\Rightarrow (D+1)Y = \frac{\pi}{4} e^{-x}$

$Y' + Y = \frac{\pi}{4} e^{-x}$

$dY + Y dx = \frac{\pi}{4} e^{-x} dx$

$e^{x} dY + Y e^{x} dx = \frac{\pi}{4} dx \Leftarrow$ EXACT!

$\Rightarrow Y e^{x} = \frac{\pi}{4} x$

$Y = \frac{\pi}{4} x e^{-x}$

$\Rightarrow Y = Ae^{-x} + Be^{x} + \frac{\pi}{4} x e^{-x}$

(CONT)

BOUNDRY CONDITIONS

$$I(0) = \int_0^\infty \frac{1}{(1+x^2)^2}\,dx$$

$$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$$

$$x = 0 \Rightarrow \theta = 0, \quad x = \infty \Rightarrow \theta = \frac{\pi}{2}$$

$$I(0) = \int_0^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[1 + \cos 2\theta\right] d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

$$I(a) = Ae^{-a} + Be^{a} + \frac{\pi}{4} a e^{-a}$$

$$I(0) = A + B = \pi/4$$

ALSO $I(\infty) = 0 \Rightarrow B = 0 \Rightarrow A = \frac{\pi}{4}$

THUS

$$I(a) = \int_0^\infty \frac{\cos ax}{(1+x^2)^2}\,dx = \frac{\pi}{4}e^{-a} + \frac{\pi}{4} a e^{-a}$$

$$= (a+1)\frac{\pi}{4}e^{-a}$$

3-5. $I = \int d^3x \; e^{i\vec{a}\cdot\vec{x}} e^{-br^2}$

CHOOSE $\vec{x}$ IN DIRECTION OF $\vec{a} \Rightarrow \vec{a}\cdot\vec{x} = ax$

$I = \int d^3x \; e^{iax} e^{-br^2}$

$r^2 = x^2 + y^2 + z^2$

$\Rightarrow I = \int_{-\infty}^{\infty} e^{-by^2} dy \int_{-\infty}^{\infty} e^{-bz^2} dz \int_{-\infty}^{\infty} e^{iax - bx^2} dx$

NOW $\int_{-\infty}^{\infty} e^{-b\xi^2} d\xi = \sqrt{\frac{\pi}{b}}$

$\Rightarrow I = \frac{\pi}{b} \int_{-\infty}^{\infty} e^{-[bx^2 + iax]} dx$

$= \frac{\pi}{b} \int_{-\infty}^{\infty} e^{-b[x^2 + \frac{ia}{b}x]} dx$

$= \frac{\pi}{b} \int_{-\infty}^{\infty} e^{-b(x + \frac{ia}{2b})^2} e^{(\frac{a}{2b})^2} dx$

$= \frac{\pi}{b} e^{(\frac{a}{2b})^2} \int_{-\infty}^{\infty} e^{-b(x + \frac{ia}{2b})^2} dx$

$= \frac{\pi}{b} e^{+(\frac{a}{2b})^2} \int_{-\infty}^{\infty} e^{-b\xi^2} d\xi$

$= \frac{\pi}{b} e^{(\frac{a}{2b})^2} \sqrt{\frac{\pi}{b}}$

$= \left(\frac{\pi}{b}\right)^{3/2} e^{a^2/4b^2}$

3-6. $I = \int d^3x \; \vec{x} \, e^{i\vec{a}\cdot\vec{x}} e^{-br^2}$

CHOOSE $\vec{x}$ IN DIRECTION OF $\vec{a}$, LET $\hat{a} = \frac{\vec{a}}{|a|}$

$\Rightarrow I = \int d^3x \; x \, e^{iax} e^{-br^2} \hat{a}$

FROM 3-5: $\int d^3x \, e^{iax} e^{-br^2} = \left(\frac{\pi}{b}\right)^{3/2} e^{a^2/4b^2}$

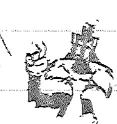
$\frac{d}{da} \int d^3x \, e^{-iax} e^{-br^2} = \int d^3x \, (-ix) e^{-iax} e^{-br^2}$

$= -i|I| = \frac{d}{da} \left(\frac{\pi}{b}\right)^{3/2} e^{a^2/4b^2}$

$= \left(\frac{\pi}{b}\right)^{3/2} \frac{2a}{4b^2} e^{a^2/4b^2}$

$= \left(\frac{\pi}{b}\right)^{3/2} \frac{a}{2b^2} e^{a^2/4b^2}$

$\Rightarrow |I| = i \left(\frac{\pi}{b}\right)^{3/2} \frac{a}{2b^2} e^{a^2/4b^2}$

AND $I = i \left(\frac{\pi}{b}\right)^{3/2} \frac{a}{2b^2} e^{a^2/4b^2} \hat{a}$ 

3-7. $I = \int_0^1 \frac{1}{x} \ln\left(\frac{1+x}{1-x}\right) dx$

$= \int_0^1 \frac{1}{x}\left[\ln(1+x) - \ln(1-x)\right] dx$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + -$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - -$

$\Rightarrow I = \int_0^1 \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right] dx$

$= \int_0^1 \frac{2}{x}\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right] dx$

$= \int_0^1 2\left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots\right] dx$

DEFINE:

$I(\alpha) = \int_0^1 2\left[\alpha + \frac{x^2\alpha^3}{3} + \frac{x^4\alpha^5}{5} + \frac{x^6\alpha^7}{7} + \dots\right] dx$

$I'(\alpha) = \int_0^1 2\left[1 + (x\alpha)^2 + (x\alpha)^4 + (x\alpha)^6 + \dots\right] dx$

$= \int_0^1 \frac{2\,dx}{1-(x\alpha)^2}$

SKIP $= \frac{1}{\alpha}\int_0^\alpha \frac{2\,d(x\alpha)}{1-(x\alpha)^2} = \frac{1}{\alpha}\int_0^\alpha \frac{2\,d\xi}{1-\xi^2}$ SKIP

LET $\xi = \sin\theta$

$d\xi = \cos\theta\, d\theta$

$\xi = 0 \Rightarrow \theta = 0 \quad \xi = \alpha \Rightarrow \theta = \sin^{-1}\alpha$

$I'(\alpha) = \frac{2}{\alpha}\int_0^{\sin^{-1}\alpha} \frac{\cos\theta\, d\theta}{\cos^2\theta}$

$= \frac{2}{\alpha}\int_0^{\sin^{-1}\alpha} \frac{d\theta}{\cos\theta}$ $\Leftarrow$ ARG!

TRY INSTEAD, DIRECT EVALUATION:

$I'(\alpha) = \frac{2}{\alpha}\int_0^\alpha \frac{d\xi}{1-\xi^2}$

$= \frac{2}{\alpha} \left.\frac{\xi}{1-\xi}\right|_0^\alpha$ $\Leftarrow$ Pg. 50

$= \frac{2}{\alpha}\frac{\alpha}{1-\alpha}$

$= \frac{2}{1-\alpha}$

$I(\alpha) = -2\ln(1-\alpha) + C$

$I(0) = 0 \Rightarrow C = 0$

$\uparrow\uparrow$ $I = I(1) = \infty$ NO! $\uparrow\uparrow$

SKIP SKIP

(CONT):

$$I'(\alpha) = \int_0^1 \frac{2\,dx}{1 - (x\alpha)^2}$$

$$= \frac{2}{\alpha^2} \int_0^1 \frac{dx}{1/\alpha^2 - x^2}$$

$$= \frac{2}{\alpha^2} \left[\frac{\alpha^2}{2} \ln \frac{\frac{1}{\alpha} + x}{\frac{1}{\alpha} - x} \right]_0^1$$

$$= \ln \frac{1+\alpha x}{1-\alpha x} \Big|_0^1$$

$$= \ln \frac{1+\alpha}{1-\alpha}$$

$$= \ln(1+\alpha) - \ln(1-\alpha)$$

$$I(\alpha) = \int^\alpha I'(\alpha)\,d\alpha$$

$$= \int^\alpha [\ln(1+\alpha) - \ln(1-\alpha)]\,d\alpha$$

NOW: $\int \ln\alpha\,d\alpha = \alpha \ln\alpha - \alpha$

$$\Rightarrow \int \ln(\alpha+1)\,d\alpha = (\alpha+1)\ln(\alpha+1) - (\alpha+1)$$

$$- \int \ln(1-\alpha)\,d\alpha = (1-\alpha)\ln(1-\alpha) - (1-\alpha)$$

ADD 'EM UP:

$$I(\alpha) = (1+\alpha)\ln(\alpha+1) + (1-\alpha)\ln(1-\alpha) - 2 + C$$

$$I(0) = 0 \Rightarrow C = +2$$

$$\Rightarrow I(\alpha) = (1+\alpha)\ln(\alpha+1) + (1-\alpha)\ln(1-\alpha)$$

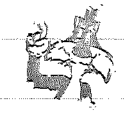
$$I = I(1)$$

$$= 2\ln 2 + \lim_{\alpha \to 1} \frac{\ln(1-\alpha)}{(1-\alpha)^{-1}}$$

USE L'HOPITAL'S:

$$I = 2\ln 2 + \lim_{\alpha \to 1} \frac{\frac{-1}{1-\alpha}}{\frac{-1}{(1-\alpha)^2}}$$

$$= 2\ln 2 + \lim_{\alpha \to 1} (1-\alpha)$$



$$= 2\ln 2$$

3-8. $\int_0^\infty \frac{dx}{\cosh x} = \int_0^\infty \operatorname{sech} x\,dx$

$$= \tan^{-1} \sinh x \Big|_0^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} 0$$

$$= \frac{\pi}{2} - 0$$

$$= \pi/2$$

$$3\text{-}9.\ I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\int_0^{\infty} \frac{dx}{(1+x^2)^2}$$

$$x = \tan\theta$$

$$dx = \sec^2\theta\, d\theta$$

$$x = 0 \Rightarrow \theta = 0,\quad x = \infty \Rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow I = 2\int_0^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\, dx$$

$$= 2\int_0^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

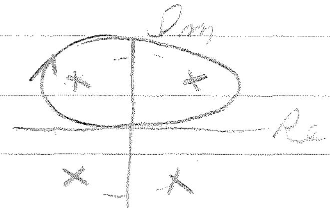
$$= 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= \int_0^{\pi/2} [1 + \cos 2\theta]\, d\theta$$

$$= \pi/2$$

$3\text{-}10.\ I = \int_0^{\infty} \frac{dx}{x^4+1}$

$x^4+1 \Rightarrow x^4 = -1 = \begin{cases} e^{-i\pi} \Rightarrow x = e^{-i\pi/4} \\ e^{-i3\pi} \Rightarrow x = e^{-i\pi 3/4} \\ e^{i\pi} \Rightarrow x = e^{i\pi/4} \\ e^{-i\pi 3} \Rightarrow x = e^{i\pi 3/4} \end{cases}$



$$2I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = i2\pi \left[\mathrm{Res} \frac{1}{x^4+1}\Big|_{e^{i\pi/4}} + \mathrm{Res} \frac{1}{x^4+1}\Big|_{e^{i\frac{3\pi}{4}}} \right]$$

$$\mathrm{Res}\left[\frac{1}{x^4+1}\right]_{e^{i\pi/4}} = \lim_{x \to e^{i\pi/4}} \frac{1}{\frac{d}{dx}(x^4+1)}$$

$$= \lim_{x \to e^{i\pi/4}} \frac{1}{4x^3} = \frac{1}{4} e^{-i3\pi/4}$$

$$= -\frac{1}{4} e^{i\pi/4}$$

$$\mathrm{Res}\left[\frac{1}{x^4-1}\right]_{e^{i3\pi/4}} = \frac{1}{4} e^{-i9\pi/4} = \frac{1}{4} e^{-i\pi/4}$$

$$\Rightarrow 2I = -i2\pi \left[\frac{1}{4} e^{i\pi/4} - \frac{1}{4} e^{-i\pi/4} \right]$$

$$= \pi \frac{e^{i\pi/4} - e^{-i\pi/4}}{i2}$$

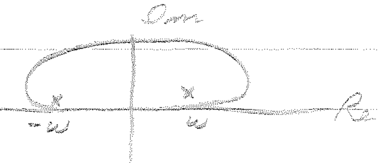
$$= \pi \sin \frac{\pi}{4}$$

$$= \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}\,\pi}{4}$$

$$3\text{-}11.\ I = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2 - \omega_0^2}\, d\omega = \int_{-\infty}^{\infty} \frac{\cos \omega t}{\omega^2 - \omega_0^2}\, d\omega$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - \omega_0)(\omega + \omega_0)}\, d\omega$$



$$\mathrm{Res}\ \frac{e^{i\omega t}}{(\omega-\omega_0)(\omega+\omega_0)}\Big|_{\omega_0} = \lim_{\omega \to \omega_0} \frac{e^{i\omega t}}{\omega + \omega_0} = \frac{e^{i\omega_0 t}}{2\omega_0}$$

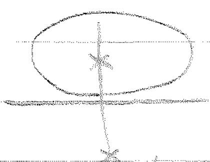
$$\mathrm{Res}\ \frac{e^{i\omega t}}{(\omega-\omega_0)(\omega+\omega_0)}\Big|_{\omega_0} = \frac{e^{-i\omega_0 t}}{-2\omega_0}$$

$$I = i\,2\pi \left[\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega_0 t}}{2\omega_0} \right]$$

$$= -\frac{2\pi}{\omega_0} \sin \omega_0 t$$

3-12 $I = \int_{-\infty}^{\infty} \frac{x^2\,dx}{(a^2+x^2)^2}$

$$= \int_{-\infty}^{\infty} \frac{x^2\,dx}{[(x+ia)(x-ia)]^2}$$



$$I = \int_c \frac{z^2\,dz}{(a^2+z^2)^2} = \int_c f(z)\,dz$$

$$\mathrm{Res}\, f(z)\Big|_{ia} = (1)!(z-ia)^2 \frac{d}{dz} f(z)\Big|_{ia}$$

$$= (z-ia)^2 \frac{2(a^2+z^2)^2 z - z^2 \frac{d}{dz}(a^2+z^2)^2}{(x+ia)^4(x-ia)^4}$$

$$= \frac{2(a^2+z^2)^2 z - z^2(2z)\,2(a^2+z^2)}{(x+ia)^4(x-ia)^2}$$

$$= \frac{2z\cancel{(a^2+z^2)}(a^2+z^2-2z^2)}{(z+ia)^2\,\cancel{(x^2+z^2)}}$$

$$= \frac{2z(a^2-z^2)}{(z+ia)^2}\Big|_{z=ia}$$

$$= \frac{i2a(a^2+a^2)}{(i2a)^2}$$

$$= \frac{i2a^3}{-4a^2}$$

$$= \frac{a}{2i}$$

$$I = i2\pi\left(\frac{a}{i2}\right) = a\pi$$

3-13. $I = \int \frac{d^3r}{(a^2+r^2)^3}$

$$\int f\,d^3r = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \int_0^{\infty} r^2 f\,dr$$

IF f IS ϕ AND θ INDEPENDENT:

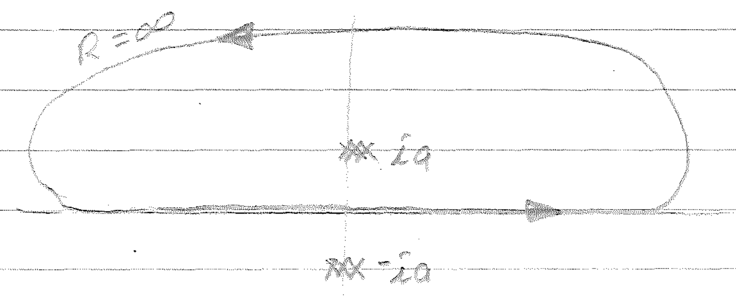
$$\int f\,d^3r = 4\pi \int_0^{\infty} r^2 f\,dr$$

THUS:

$$I = 4\pi \int_0^{\infty} \frac{r^2 dr}{(a^2+r^2)^3}$$

$$= 2\pi \int_{-\infty}^{\infty} \frac{r^2}{(a^2+r^2)^3}$$

$$= 2\pi \int_{-\infty}^{\infty} \frac{r^2\,dr}{(r+ia)^3(r-ia)^3}$$



$$f(z) = 2\pi \frac{z^2}{(a^2+z^2)^3} = \frac{2\pi z^2}{(z+ia)^3(z-ia)^3}$$

$$I = \int_C f(z)\,dz = i\,2\pi\,\mathrm{Res}\, f(z)\big|_{ia}$$

(CONT)

ia IS THIRD ORDER POLE $(n=3)$

$$\operatorname{Res} f(z) = \frac{1}{2!}\frac{d^2}{dz^2}(z-ia)^3 f(z)\Big|_{ia}$$

$$= \frac{2\pi}{2}\frac{d^2}{dz^2}\frac{z^2}{(z+ia)^3}\Big|_{ia}$$

$$= \pi\frac{d}{dz}\frac{(z+ia)^{\cancel{3}\,1}\,2z - z^2\cdot 3\cancel{(z+ia)^2}}{(z+ia)^{\cancel{6}\to 4}}\Big|_{ia}$$

$$= \pi\frac{d}{dz}\frac{2z(z+ia)-3z^2}{(z+ia)^4}\Big|_{ia}$$

$$= \pi\frac{d}{dz}\frac{-z^2+i2za}{(z+ia)^4}\Big|_{ia}$$

$$= \pi\frac{d}{dz}\frac{i2za-z^2}{(z+ia)^4}\Big|_{ia}$$

$$= \pi\frac{\cancel{(z+ia)^4(i2a}-2z)^{0} - z(i2a-z)4(z+ia)^3}{(z+ia)^8}\Big|_{ia}$$

$$= -\pi\frac{z(i2a-z)}{(z+ia)^5}\Big|_{ia}$$

$$= -\pi\frac{ia(ia)}{(i2a)^5} = \frac{-\pi}{2^5}\frac{(ia)^2}{(ia)^5} = \frac{-\pi}{2^5}\frac{1}{(ia)^3}$$

$$= \frac{-\pi}{2^5}\frac{1}{-ia^3}$$

$$= \frac{\pi}{i2^5a^3}$$

$$\Rightarrow I = \cancel{i}2\pi\times\frac{\pi}{\cancel{i}2^5a^3} = \frac{\pi^2}{2^4a^3} = \frac{\pi^2}{16a^3}$$

3.14. $I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}\,(a+bx)}\,dx \quad ; \; a>b>0$

LET $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$x = -1 \Rightarrow \theta = \pi$

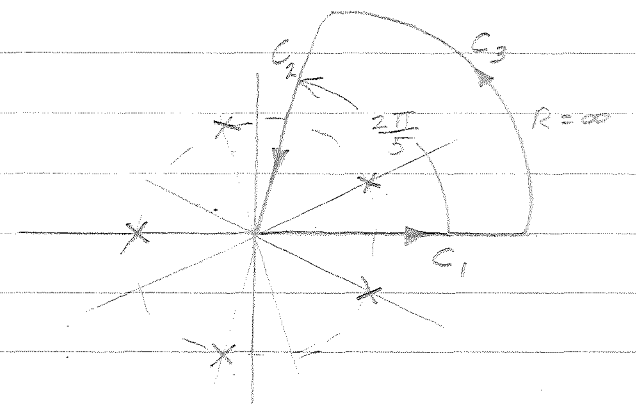
$x = 1 \Rightarrow \theta = 0$

$\therefore I = \int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta}\,(a+b\cos\theta)}$

$= \int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$ $\Leftarrow$ WORKED ON pp. 69-70

$= \frac{\pi}{\sqrt{a^2-b^2}} \quad ; \; a>b>0$

3-15. $I = \int_0^\infty \frac{x\,dx}{1+x^5}$



POLES @ $x^5 = -1$

1. $e^{i\pi/5}$
2. $e^{i3\pi/5} = -e^{-i2\pi/5}$
3. -1
4. $e^{i7\pi/5} = -e^{i2\pi/5}$
5. $e^{i9\pi/5} = e^{-i\pi/5}$

$$f(z) = \frac{z}{1+z^5}$$

$$= \frac{z}{(z-e^{i\pi/5})(z+e^{-i2\pi/5})(z+1)(z+e^{i2\pi/5})(z-e^{-i\pi/5})}$$

$$\int_C f(z)\,dz = \int_{C_1} f(z) + \int_{C_3} f(z) + \int_{C_2} f(z)$$

$$I = \int_{C_1} f(z)\,dz \qquad \int_{C_3} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz = \int_{\infty+i\infty}^{0} \frac{z}{1+z^5}\Big|_{z=re^{i\frac{2\pi}{5}}} dz$$

$$= -\int_0^\infty \frac{re^{i2\pi/5}}{1+r^5}\,dr \times e^{i\frac{2\pi}{5}}$$

$$= -e^{i4\pi/5}\, I$$

THUS

$$\int_C f(z)\,dz = (1 - e^{i4\pi/5})\, I$$

(CONT)

$$\operatorname{Res} f(z)\Big|_{e^{i\pi/5}} = \frac{e^{i\pi/5}}{(e^{i\pi/5}-e^{i3\pi/5})(e^{i\pi/5}+1)(e^{i\pi/5}-e^{i7\pi/5})(e^{i\pi/5}-e^{i9\pi/5})}$$

$$= \frac{\cancel{e^{i\pi/5}}}{\cancel{e^{i\pi/5}}(1-e^{i2\pi/5})\, e^{i\pi/10}(e^{i\pi/10}+e^{-i\pi/10})\, e^{i\pi/5}(1-e^{i6\pi/5})}$$

$$\times \frac{1}{e^{i\pi/5}(1-e^{i8\pi/5})}$$

$$= \frac{+1}{e^{i\pi/2}(e^{i2\pi/5}-1)\, 2\cos\frac{\pi}{10}\,(e^{i6\pi/5}-1)(e^{i8\pi/5}-1)}$$

$$= \frac{+1}{i2e^{i\pi/5}(e^{i\pi/5}-e^{-i\pi/5})\cos\frac{\pi}{10}\, e^{i3\pi/5}(e^{i3\pi/5}-e^{-i3\pi/5})}$$

$$\times \frac{+1}{e^{i4\pi/5}(e^{i4\pi/5}-e^{-i4\pi/5})}$$

$$= \frac{+1}{i2\, e^{i8\pi/5}(i2\sin\frac{\pi}{5})(\cos\frac{\pi}{10})(i2\sin\frac{3\pi}{5})(i2\sin\frac{4\pi}{5})}$$

$$= \frac{+1}{8\, e^{-i2\pi/5}\sin\frac{\pi}{5}\cos\frac{\pi}{10}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}}$$

$$\sin\frac{3\pi}{5} = +\cos\frac{\pi}{5} \qquad \sin\frac{4\pi}{5} = \sin\frac{\pi}{5}$$

$$\Rightarrow \operatorname{Res} f(z)\Big|_{e^{i\pi/5}} = \frac{+1}{8\, e^{i2\pi/5}\sin^2\frac{\pi}{5}\cos\frac{\pi}{10}\cos\frac{\pi}{5}}$$

$$\int_C f(z)\,dz = \frac{+i\pi}{4\, e^{-i2\pi/5}\sin^2\frac{\pi}{5}\cos\frac{\pi}{5}\cos\frac{\pi}{10}}$$

(CONT.)

$$I = \frac{i\pi}{4\sin^2\frac{\pi}{5}\cos\frac{\pi}{5}\cos{\pi/10}} \times \frac{e^{i2\pi/5}}{1 - e^{i4\pi/5}}$$

$$i\,e^{i2\pi/5} = e^{i7\pi/5} = e^{-i3\pi/5}$$

$$e^{i4\pi/5} = e^{-i\pi/5}$$

$$I = \left(\frac{\pi}{4\sin^2\frac{\pi}{5}\cos\frac{\pi}{5}\cos\frac{\pi}{10}}\right)\frac{e^{-i3\pi/5}}{1 - e^{-i\pi/5}}$$

$$= (\quad)\ \frac{e^{-i3\pi/5}(1 - e^{i\pi/5})}{(1 - e^{+i\pi/5})(1 - e^{-i\pi/5})}$$

$$= (\quad)\ \frac{e^{-i3\pi/5} - e^{-i2\pi/5}}{1 - (e^{i\pi/5} + e^{-i\pi/5}) + 1}$$

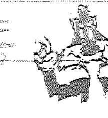
$$= (\quad)\ \frac{-\left[e^{i2\pi/5} + e^{-i2\pi/5}\right]}{2\left[1 - \cos{\pi/5}\right]}$$

$$= (\quad)\ \frac{\cos{\pi/5}}{1 - \cos{\pi/5}}$$

$$= \frac{\pi}{4\sin^2\frac{\pi}{5}\cos\frac{\pi}{10}\,(1 - \cos{\pi/5})}$$

$$1 - \cos{\pi/5} = 2\sin^2\frac{\pi}{10}$$

$$\Rightarrow I = \frac{\pi}{8\sin^2\frac{\pi}{5}\cos\frac{\pi}{10}\sin^2\frac{\pi}{10}}$$



3-16. $I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta$

$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta \Rightarrow \frac{dz}{iz} = d\theta$$

$$\sin\theta = \frac{1}{i2}\left(z - \frac{1}{z}\right)$$

$$\sin^2\theta = -\frac{1}{4}\left[z^2 - 2 + \frac{1}{z^2}\right]$$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] \qquad ;\ C = \text{UNIT CIRCLE}$$

$$I = \int_C \frac{-\frac{1}{4}\left[z^2 - 2 + \frac{1}{z^2}\right]}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{i4}\int_C \frac{z^2 - 2 + 1/z^2}{\frac{b}{2}z^2 + az + b/2}\, dz$$

$$= -\frac{1}{i2}\int_C \frac{z^2 - 2 + 1/z^2}{bz^2 + 2az + b}\, dz$$

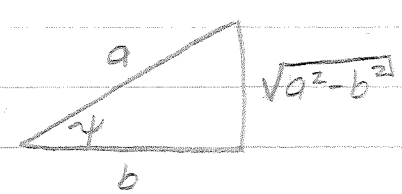
$$= \frac{i}{2}\int_C \frac{z^4 - 2z^2 + 1}{z^2(bz^2 + 2az + b)}\, dz$$

$$= \frac{i}{2}\int_C \frac{(z^2 - 1)^2}{z^2(bz^2 + 2az + b)}\, dz$$

WE NOW GOTTA FIGURE OUT WHICH (IF ANY) OF THE ROOTS OF THE DENOMINATOR'S QUADRADIC LIE WITHIN THE UNIT CIRCLE (GIVEN THAT $a < |b| < 0$

$$z_r = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

TO FIGGER IT OUT, DEFINE $\psi \ni$



THEN $\quad -\frac{a}{b} \pm \frac{\sqrt{a^2-b^2}}{b} = \frac{-1}{\cos\psi} \pm \tan\psi$

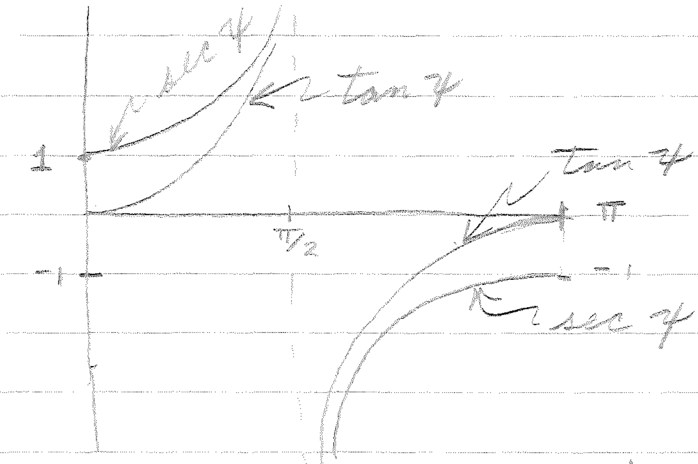
TO COVER ALL BASES, WE MUST CONSIDER $0 < \psi < \pi$.

WE WISH, THEN, TO DETERMINE WHEN

$$|\sec \psi \pm \tan \psi| < 1 \quad ; 0 < \psi < \pi$$

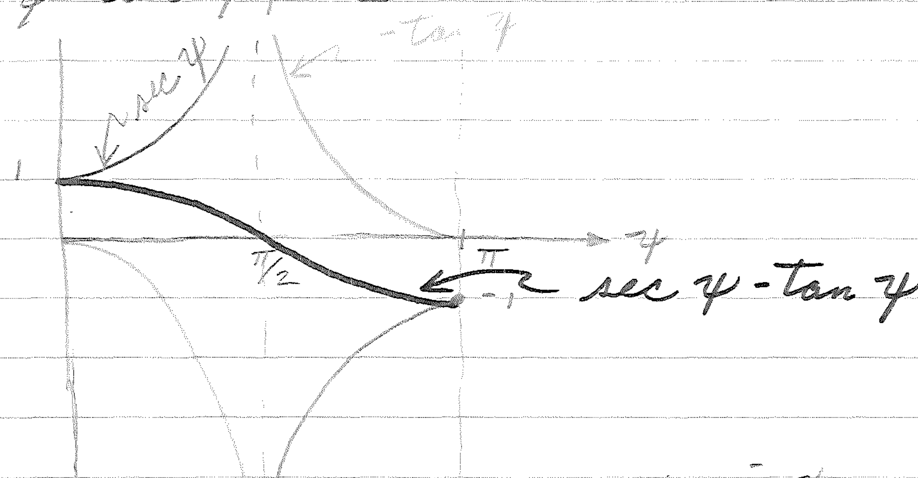
① $|\sec \psi + \tan \psi| < 1$

CONSIDER THE SKETCHES



OBVIOUSLY, $|\sec \psi + \tan \psi| \geq 1 \quad \forall \psi$

② $|\sec \psi - \tan \psi| < 1$



$$\lim_{\psi \to \pi/2} \sec \psi - \tan \psi = \lim_{\psi \to \pi/2} \frac{1 - \sin \psi}{\cos \psi}$$

$$= \lim_{\psi \to \pi/2} \frac{+\cos \psi}{\sin \psi} = 0$$

$$\therefore |\sec \psi - \tan \psi| \leq 1$$

OK. LOTS OF TROUBLE TO SHOW THAT,

GIVEN $a < |b| < 0$:

$$\begin{cases} ① \left| -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} \right| > 1 \\ ② \left| -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} \right| < 1 \end{cases}$$

ONWARD:

$$I = \frac{i}{2} \int_c \frac{(z^2-1)^2}{z^2(bz^2+2az+b)}$$

$$f(z) = \frac{i(z^2-1)^2}{2z^2(bz^2+2az+b)} = \frac{i(z^2-1)^2}{2z^2\left[z+\frac{a+\sqrt{a^2-b^2}}{b}\right]\left[z+\frac{a-\sqrt{a^2-b^2}}{b}\right]}$$

① $\operatorname{Res} f(z)\big|_{z=a}$

$$= \frac{1}{1!} \frac{d}{dz} \frac{i(z^2-1)^2}{2(bz^2+2az+b)}\bigg|_{z=0}$$

$$= \frac{i}{2}\left[\frac{(bz^2+2az+b)(z^2-1)2z-(z^2-1)^2(2bz+2a)}{(bz^2+2az+b)^2}\right]\Bigg|_{z=0}$$

$$= \frac{i}{2}\,\frac{-(-1)^2(2a)}{b^2} = \frac{-ia}{b^2}$$

② $\operatorname{Res} f(z)\big|_{z=-\frac{a}{b}+\frac{\sqrt{a^2-b^2}}{b}}$

$$= \frac{i}{2}\,\frac{(z^2-1)^2}{z^2\left[z+\frac{a}{b}+\frac{\sqrt{a^2-b^2}}{b}\right]}\Bigg|_{\frac{-a+\sqrt{a^2-b^2}}{b}}$$

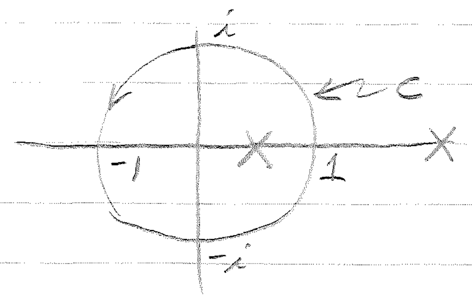
$$= \frac{i}{2}\,\frac{\left[\frac{\left[-a+\sqrt{a^2-b^2}\right]^2-b^2}{b^2}\right]^2}{\frac{\left(-a+\sqrt{a^2-b^2}\right)^2}{b^2}\left[\frac{2\sqrt{a^2-b^2}}{b}\right]}$$

$$= \frac{i}{4b}\,\frac{\left[\left(-a+\sqrt{a^2-b^2}\right)^2-b^2\right]^2}{\left(-a+\sqrt{a^2-b^2}\right)^2\sqrt{a^2-b^2}}$$

$$= \frac{i}{4b}\,\frac{\left[\left(\sqrt{a^2-b^2}-a\right)^2-b^2\right]^2}{\left(\sqrt{a^2-b^2}-a\right)^2\sqrt{a^2-b^2}}$$

LETS LEAVE IT IN THIS FORM, SINCE IT DOESN'T SEEM TO SIMPLIFY ANY.

(CONT)



$$\int_C f(z)\,dz = i2\pi \sum \operatorname{Res} f(z)$$

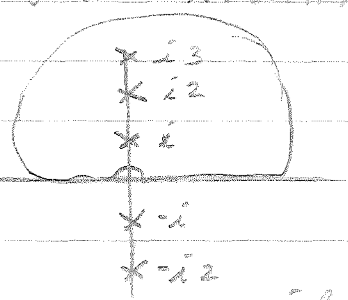
$$= i2\pi\left[\frac{-ia}{2b^2} + \frac{i}{4b}\,\frac{(d^2-b^2)^2}{d^2\sqrt{a^2-b^2}}\right];\ d = \sqrt{a^2-b^2} - a$$

$$= \frac{+a\pi}{b^2} - \frac{\pi}{2b}\,\frac{(d^2-b^2)^2}{d^2\sqrt{a^2-b^2}} = I$$

$$\Rightarrow I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{a\pi}{b^2} - \frac{\pi}{2b}\,\frac{(d^2-b^2)^2}{d^2\sqrt{a^2-b^2}}\,;$$

$$\begin{cases} a > |b| > 0 \\ d = \sqrt{a^2-b^2} - a \end{cases}$$

3-17. $I = \int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx$



NOTE: $\frac{\sinh ax}{\sinh \pi x} \xrightarrow[x \to \infty]{} e^{(a-\pi)x}$

$\Rightarrow I = \infty$ FOR $a > \pi$

ALSO $\lim_{x \to 0} \frac{\sinh ax}{\sinh \pi x} = \frac{a}{\pi}$

$f(z) = \frac{\sinh az}{\sinh \pi z}$

$$\operatorname{Res} f(z)\Big|_{z=ni} = \lim_{z \to in} (z - in) \frac{\sinh az}{\sinh \pi z}$$

$$= \lim_{z \to in} (z - in) \frac{\sin iaz}{\sin i\pi z}$$

$$= \lim_{z \to in} \frac{(z-in)\sin iaz}{\sin i\pi z}$$

USE LAHOPITAL:

$$= \lim_{z \to in} \frac{ia(z-in)\cos iaz + \sin iaz}{i\pi \cos i\pi z}$$

$$= \frac{\sin ia(in)}{i\pi \cos i\pi(in)} = \frac{i \sin an\pi}{\pi \cos \pi n}$$

$$= \frac{-1}{i\pi} \sin(a\pi n)(-1)^n$$

$$I = i2\pi\left(\frac{-1}{i\pi}\right)\sum_{n=1}^{\infty} \sin an\,(-1)^n$$

$$= -2 \operatorname{Im} \sum_1 (-1)^n e^{ian}$$

$$Y = e^{ia}$$

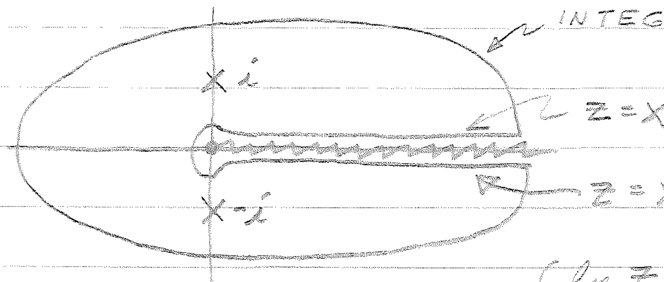
$$I = -2 \operatorname{Im} \sum_1 (-1)^n Y^n = \operatorname{Im} \frac{-2}{1+Y} = \operatorname{Im} \frac{-2}{1+e^{ia}}$$

$$= -2 \operatorname{Im} \frac{1+e^{-ia}}{(1+e^{ia})(1+e^{-ia})} = -2 \operatorname{Im} \frac{1+e^{-ia}}{2+e^{ia}+e^{-ia}}$$

$$= -2 \operatorname{Im} \frac{1+\cos a - i\sin a}{2[1+\cos a]}$$

$$= \begin{cases} \frac{\sin a}{1+\cos a} & ; a < \pi \\ \infty & ; a \geq \pi \end{cases}$$

$$3\text{-}18.\quad I = \int_0^\infty \frac{[\ln x]^2}{x^2+1}\,dx$$



$; \; z^2 = x^2$

CONSIDER $\quad f(z) = \dfrac{(\ln z)^3}{z^2+1} = \dfrac{(\ln z)^3}{(z+i)(z-i)}$

$$\int_C f(z)\,dz = \int_0^\infty \frac{(\ln x)^3\,dx}{x^2+1} - \int_0^\infty \frac{(\ln x + i2\pi)^3}{x^2+1}\,dx$$

$$= \int_0^\infty \frac{(\ln x)^3\,dx}{x^2+1} - \int_0^\infty \frac{(\ln x)^3}{x^2+1}\,dx$$

$$- 3(i2\pi)\int_0^\infty \frac{(\ln x)^2\,dx}{x^2+1} - 3(i2\pi)^2\int_0^\infty \frac{\ln x\,dx}{x^2+1}$$

$$- (i2\pi)^3 \int_0^\infty \frac{dx}{x^2+1}$$

$$= -i6\pi I + 12\pi^2 J + i8\pi^3 \int_0^\infty \frac{dx}{x^2+1}$$

WHERE $\quad J = \displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx$

FROM Pg 66, 77 ; $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$

$$\therefore \int_C f(z) = -i6\pi I + 12\pi^2 J + i8\pi^3\left(\frac{\pi}{2}\right)$$

$$= -i6\pi I + 12\pi^2 J + i4\pi^4$$

TO FIND J, CONSIDER $\quad g(z) = \dfrac{(\ln z)^2}{z^2+1}$

$$\Rightarrow \int_C g(z)\,dz = \int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx - \int_0^\infty \frac{(\ln x + i2\pi)^2}{x^2+1}\,dx$$

$$= \int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx - \int_0^\infty \frac{(\ln x)^2}{x^2+1}$$

$$- 2(i2\pi)\int_0^\infty \frac{\ln x}{x^2+1}\,dx - (i2\pi)^2 \int_0^\infty \frac{dx}{x^2+1}$$

$$= -i4\pi J + 4\pi^2 \int_0^\infty \frac{dx}{1+x^2}$$

$$= -i4\pi J + 4\pi^2\left(\frac{\pi}{2}\right)$$

$$= -i4\pi J + 2\pi^3$$

(CONT.)

$$\operatorname{Res} g(z)\big|_{z=i} = \frac{(\ln i)^2}{i2} = \frac{(\ln e^{i\pi/2})^2}{i2} = (i\pi/2)^2/i2 = -\frac{\pi^2}{i8}$$

$$\operatorname{Res} g(z)\big|_{z=-i} = \frac{(\ln(-i))^2}{-i2} = \frac{(\ln e^{-i\pi/2})^2}{-i2} = \frac{\pi^2}{i8}$$

$$\therefore \int_C g(z)\,dz = 0$$

THUS $\quad i4\pi J = 2\pi^3$

$$J = \frac{2\pi^3}{i4\pi} = \frac{\pi^2}{i2} = -i\frac{\pi^2}{2}$$

AND $\quad \int_C f(z)\,dz = -i6\pi I + 12\pi^2 J + i4\pi^4$

$$= i6\pi I - i6\pi^4 + i4\pi^4$$

$$= i6\pi I - i2\pi^4$$

$$\operatorname{Res} f(z)\big|_{z=i} = \frac{(\ln i)^3}{i2} = \frac{(\ln e^{i\pi/2})^3}{i2} = \frac{\left(\frac{i\pi}{2}\right)^3}{i2} = \frac{-i\pi^3}{i16} = -\frac{\pi^3}{16}$$

$$\operatorname{Res} f(z)\big|_{z=-i} = \frac{(\ln(-i))^3}{-i2} = \frac{(\ln e^{-i\pi/2})^3}{-i2} = \frac{(-i\pi/2)^3}{-i2} = \frac{i\pi^3}{-i16} = -\frac{\pi^3}{16}$$

$$\Rightarrow \int_C f(z)\,dz = -i2\pi\left(\frac{\pi^3}{8}\right) = -\frac{i\pi^4}{4}$$

THUS $\quad i6\pi I = -\frac{i\pi^4}{4} + i2\pi^4$

$$6\pi I = \frac{7\pi^4}{4}$$

$$6I = \frac{7\pi^3}{4}$$

$$I = 7\pi^3/24$$

3-19.

$$2I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}$$

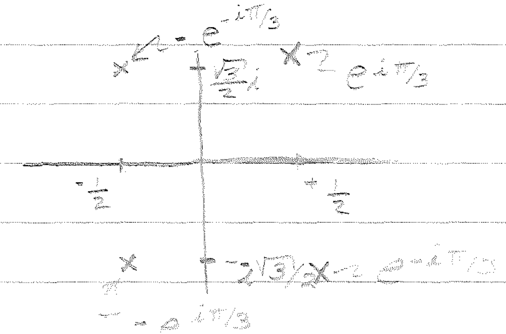
$$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2 - x^2}$$

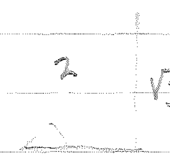
$$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+1+x)(x^2+1-x)}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+x+1)(x^2-x+1)}$$

$$x^2 + x + 1 = 0 \Rightarrow x_0 = \frac{-1 \pm i\sqrt{3}}{2} = \begin{cases} -e^{-i\pi/3} \\ -e^{i\pi/3} \end{cases}$$

$$x^2 - x + 1 = 0 \Rightarrow x_r = \frac{+1 \pm i\sqrt{3}}{2} = \begin{cases} e^{i\pi/3} \\ e^{-i\pi/3} \end{cases}$$





$$f(z) = \left[(z + e^{-i\pi/3})(z + e^{i\pi/3})(z - e^{i\pi/3})(z - e^{-i\pi/3})\right]^{-1}$$

$$\operatorname{Res} f(z)\Big|_{-e^{i\pi/3}} = \left[(-e^{-i\pi/3} + e^{i\pi/3})(-e^{-i\pi/3} - e^{i\pi/3})(-e^{-i\pi/3} - e^{i\pi/3})\right]^{-1}$$

$$= \left[(e^{i\pi/3} - e^{-i\pi/3})(e^{i\pi/3} + e^{-i\pi/3})\, 2e^{-i\pi/3}\right]^{-1}$$

$$= \left[i2\sin(\pi/3) \quad 2\cos(\pi/3) \times 2e^{-i\pi/3}\right]^{-1}$$

$$= \left[i8\sin(\pi/3)\cos(\pi/3)\right]^{-1} e^{i\pi/3}$$

$$= \left[i8 \frac{\sqrt{3}}{2} \frac{1}{2}\right]^{-1} e^{i\pi/3}$$

$$= \frac{1}{i2\sqrt{3}} e^{i\pi/3}$$

$$\operatorname{Res} f(z)\Big|_{e^{i\pi/3}} = \left[(e^{i\pi/3} + e^{-i\pi/3})(e^{i\pi/3} + e^{i\pi/3})(e^{i\pi/3} - e^{-i\pi/3})\right]^{-1}$$

$$= \left[2\cos\left(\frac{\pi}{3}\right) \times 2e^{i\pi/3} \times i2\sin\frac{\pi}{3}\right]^{-1}$$

$$= \left[2\left(\frac{1}{2}\right) \times 2 \times i2\frac{\sqrt{3}}{2}\right]^{-1} e^{-i\pi/3}$$

$$= \frac{1}{i2\sqrt{3}} e^{i\pi/3}$$

$$\sum \operatorname{Res}_c f(z) = \frac{1}{i2\sqrt{3}}\left[e^{i\pi/3} + e^{-i\pi/3}\right] = \frac{1}{i\sqrt{3}}\cos\frac{\pi}{3} = \frac{1}{i2}$$

$$I = \pi i \sum \operatorname{Res} = \pi i \left(\frac{1}{i2}\right) = \frac{\pi}{2}$$

3-20. $I = \int_0^\infty \frac{dx}{(a+bx^2)^3}$; $a>0,\ b>0$

$$2I = \int_{-\infty}^{\infty} \frac{dx}{(a+bx^2)^3}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\left[a\left(1+\frac{b}{a}x^2\right)\right]^3}$$

$$= \frac{1}{a^3}\int_{-\infty}^{\infty} \frac{dx}{(1+cx^2)^3} \qquad c = \frac{b}{a} > 0$$

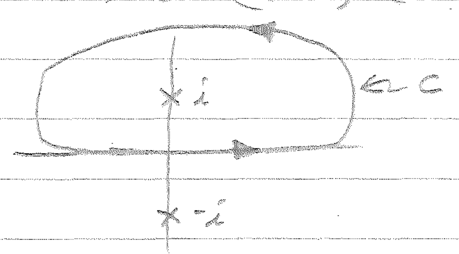
LET $\xi = \sqrt{c}\,x \Rightarrow d\xi = \sqrt{c}\,dx$

$$2I = \frac{1}{a^3}\int_{-\infty}^{\infty} \frac{1}{\sqrt{c}} \frac{d\xi}{(1+\xi^2)^3}$$

$$= \frac{1}{\sqrt{c}\,a^3}\int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^3}$$

$$J = 2\sqrt{\frac{b}{a}}\,a^3 I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{(x+i)^3(x-i)^3}$$



$$f(z) = \frac{1}{(z+i)^3(z-i)^3}$$

$$J = \int_C f(z)\,dz = i2\pi \operatorname{Res} f(z) @ i$$

$$\operatorname{Res} f(z) @ i = \frac{1}{2!}\frac{d^2}{dz^2}(z+i)^{-3}\Big|_i$$

$$= \frac{1}{2!}\frac{d}{dz}(-3)(z+i)^{-4}\Big|_i$$

$$= \frac{-3}{2}(-4)(z+i)^{-5}\Big|_i$$

$$= \frac{6}{(z+i)^5}\Big|_i$$

$$= \frac{6}{(i2)^5} = \frac{6}{i\,2^5} = \frac{3}{i\,2^4} = \frac{3}{i16}$$

(CONT)

$$\therefore\ J = 2\pi \cancel{\tfrac{1}{2}}\, \frac{3}{\cancel{4}\,16} = \frac{3\pi}{8}$$

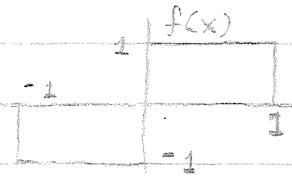
$$I = \frac{1}{2}\sqrt{\frac{a}{b}}\,\frac{1}{a^3}\,J$$

$$= \frac{1}{2}\sqrt{\frac{a}{b}}\,\frac{1}{a^3}\,\frac{3\pi}{8}$$

$$= \frac{3\pi}{16\,a^{5/2}\,b^{1/2}}$$

BOB MARKS

7-1. $f(x) = \begin{cases} 1 \ ; & 0 < x < 1 \\ -1 \ ; & -1 \le x < 0 \end{cases}$



$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$
$$= \sum_{n=0}^{\infty} \frac{2n+1}{2} \int_{-1}^{+1} P_n(\xi) f(\xi)\, d\xi \quad P_n(x)$$
$$d_n = \int_{-1}^{+1} P_n(\xi)\, f(\xi)\, d\xi = -\int_{-1}^{0} P_n(\xi)\, d\xi + \int_0^1 P_n(\xi)\, d\xi$$

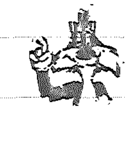
NOW $P_{2m}(\xi) = P_{2m}(-\xi)$

$$d_{2m} = -\int_{-1}^{0} P_{2m}(\psi)\, d\psi + \int_0^1 P_{2m}(\xi)\, d\xi$$
$$\psi = -\xi \Rightarrow d\psi = -d\xi$$
$$\psi = 0 \Rightarrow \xi = 0 \ ; \quad \psi = -1 \Rightarrow \xi = 1$$
$$\Rightarrow d_{2m} = -\int_1^0 P_{2m}(-\xi)(-d\xi) + \int_0^1 P_{2m}(\xi)\, d\xi$$
$$= -\int_0^1 P_{2m}(\xi)\, d\xi + \int_0^1 P_{2m}(\xi)\, d\xi = 0$$

ALSO $P_{2m+1}(\xi) = -P_{2m+1}(-\xi)$

$$d_{2m+1} = -\int_{-1}^{0} P_{2m+1}(\psi)\, d\psi + \int_0^1 P_{2m+1}(\xi)\, d\xi$$
$$\psi = -\xi$$
$$d_{2m+1} = -\int_1^0 P_{2m+1}(-\xi)(-d\xi) + \int_0^1 P_{2m+1}(\xi)\, d\xi$$
$$= -\int_0^1 P_{2m+1}(-\xi)\, d\xi + \int_0^1 P_{2m+1}(\xi)\, d\xi$$
$$= 2\int_0^1 P_{2m+1}(\xi)\, d\xi$$
$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\, d_n\, P_n(x)$$
$$= \sum_{m=0}^{\infty} \frac{2[2m+1]+1}{2} \cdot 2\int_0^1 P_{2m+1}(\xi)\, d\xi \ P_{2m+1}(x)$$
$$= \sum_{m=0}^{\infty} (4m+3) \left[\int_0^1 P_{2m+1}(\xi)\, d\xi\right] P_{2m+1}(x)$$

GOOD ENOUGH



7-2. $F(h, z) = \dfrac{1}{\sqrt{1 - 2hz + h^2}} = \sum_{n=0}^{\infty} h^n P_n(z)$

$$\int_0^h \frac{dh}{\sqrt{1-2hz+h^2}} = \sum_{n=0}^{\infty} \int_0^h h^n \, dh \; P_n(z)$$

$$= \sum_{n=0}^{\infty} \frac{h^{n+1}}{n+1} P_n(z)$$

NOW

$$\int_0^h \frac{dh}{\sqrt{1-2hz+h^2}} = \ln\left[2\sqrt{1-2hz+h^2} + 2h - 2z\right]_0^h$$

$$= \ln\left[2\sqrt{1-2hz+h^2} + 2h - 2z\right] - \ln\left[2 - 2z\right]$$

$$= \ln \frac{\sqrt{1-2hz+h^2} + (h - z)}{1 - z}$$

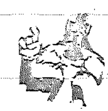
LET $h = z$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} P_n(z) = \ln \frac{\sqrt{1 - z^2}}{1 - z}$$

$$= \ln \frac{\sqrt{(1-z)(1+z)}}{1-z}$$

$$= \ln \sqrt{\frac{1+z}{1-z}}$$

$$= \frac{1}{2} \ln \frac{1+z}{1-z}$$



7-4. EVALUATE $P_n'(1)$

a. FROM RODRIGUE'S FORMULA

$$2^n n!\, P_n(x) = \left(\frac{d}{dx}\right)^n (x^2-1)^n$$

$$2^n n!\, P_n'(x) = \left(\frac{d}{dx}\right)^{n+1} (x^2-1)^n$$

$$= \left(\frac{d}{dx}\right)^{n+1} (x-1)^n (x+1)^n$$

$$= n\left(\frac{d}{dx}\right)^n \left[(x-1)^n (x+1)^{n-1} + (x-1)^{n-1}(x+1)^n\right]$$

$$\left(\frac{d}{dx}\right)^n (x-1)^n (x+1)^{n-1} \Big|_{x=1} = n!\,(x+1)^{n-1}\Big|_{x=1}$$

$$= n!\, 2^{n-1}$$

$$\left(\frac{d}{dx}\right)^n (x+1)^{n-1}(x+1)^n \Big|_{x=1} = n!\, \frac{d}{dx}(x+1)^n \Big|_{x=1}$$

$$= n\, n!\, 2^{n-1}$$

$$\Rightarrow 2^n \cancel{n!}\, P_n'(1) = n\left[n!\, 2^{n-1} + n\, n!\, 2^{n-1}\right]$$

$$= n\, \cancel{n!}\, 2^{n-1}(1+n)$$

$$P_n'(1) = \frac{1}{2} n(n+1)$$

EVALUATE $P_n'(1)$

b. USING GENERATING FUNCTION:

$$F(x,h) = \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$\frac{d}{dx}\frac{1}{\sqrt{1-2hx+h^2}} = \frac{-2h}{-2(1-2hX+h^2)^{3/2}} = \frac{h}{(1-2hx+h^2)^{3/2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} h_n P_n'(1) = \frac{h}{(1-2h+h^2)^{3/2}}$$

$$= \frac{h}{((1-h)^2)^{3/2}}$$

$$= \frac{h}{(1-h)^3}$$

CONSIDER $f(x) = \frac{1}{x^3}$

$$f(x) = \frac{1}{x^3} \qquad f(1) = 1$$

$$f'(x) = \frac{-3}{x^4} \qquad f'(1) = -3$$

$$f''(x) = \frac{3\cdot 4}{x^5} \qquad f''(1) = 3\cdot 4$$

$$f'''(x) = \frac{-3\cdot 4\cdot 5}{x^6} \qquad f'''(1) = 3\cdot 4\cdot 5$$

$\vdots$

$$f^{(n)}(x) = \frac{(n+2)!\,(-)^n}{2\,x^{n+3}} \qquad f^{(n)}(1) = \frac{(n+2)!\,(-)^n}{2}$$

$$\Rightarrow \frac{1}{x^3} = \sum_{n=0}^{\infty} \frac{(n+2)!\,(-)^n}{2\,n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)(-)^n}{2}(x-1)^n$$

$$\therefore \frac{1}{(1-h)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)(-)^n}{2}(-h)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)h^n$$

$$\frac{h}{(1-h)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)h^{n+1}$$

$$m = n+1 \Rightarrow n = m-1$$

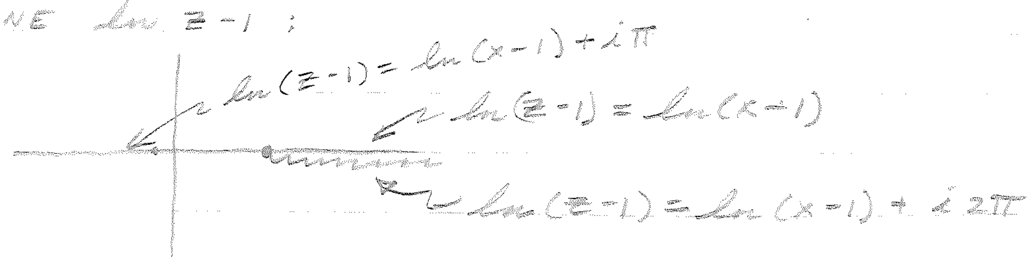
$$\frac{h}{(1-h)^3} = \frac{1}{2}\sum_{m=0}^{\infty} m(m+1)h^n = \sum_{m=0}^{\infty} h^n P_m'(1)$$

EQUATING COEFFICIENTS GIVES

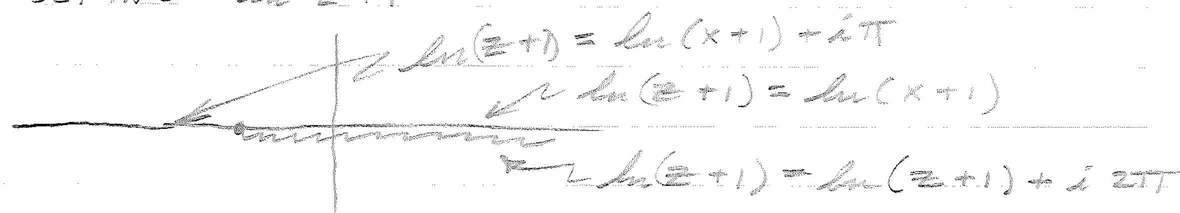
$$P_m'(1) = \frac{1}{2}m(m+1)$$

7-6. $Q_n(z) = \frac{1}{2} P_n(z) \ln\left(\frac{z+1}{z-1}\right)$

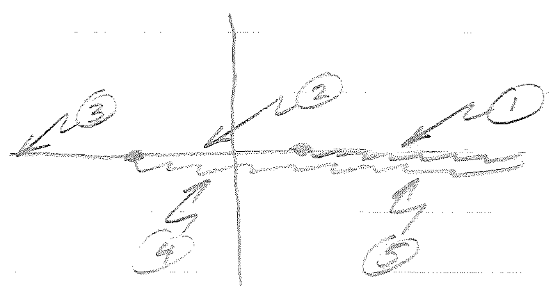
DEFINE $\ln z-1$;



DEFINE $\ln z+1$



THEN $\ln \frac{z+1}{z-1} = \ln(z+1) - \ln(z-1)$



① $\ln\left(\frac{z+1}{z-1}\right) = \ln(x+1) - \ln(x-1) = \ln \frac{x+1}{x-1}$

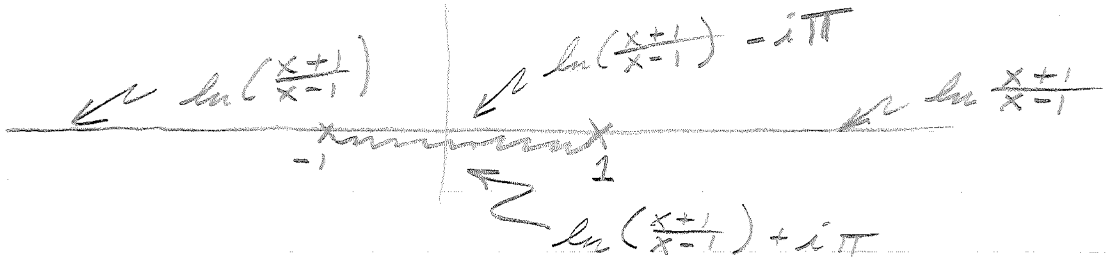
② $\ln \frac{z+1}{z-1} = \ln(x+1) - [\ln(x-1) + i\pi]$
$= \ln\left(\frac{x+1}{x-1}\right) - i\pi$

③ $\ln \frac{z+1}{z-1} = [\ln(x+1) + i\pi] - [\ln(x-1) + i\pi]$
$= \ln \frac{x+1}{x-1}$

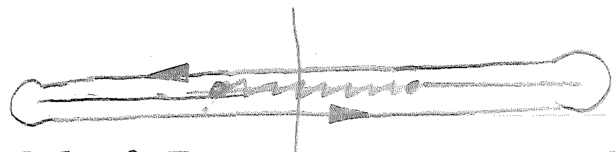
④ $\ln \frac{z+1}{z-1} = [\ln(x+1) + i\,2\pi] - [\ln(x-1) + i\pi)$
$= \ln\left(\frac{x+1}{x-1}\right) + i\pi$

⑤ $\ln \frac{z+1}{z-1} = [\ln(x+1) + i\,2\pi] - [\ln(x-1) + i\,2\pi]$
$= \ln \frac{x+1}{x-1}$

$\Rightarrow$ CUTS EAT EACH OTHER



CONSIDER INTEGRATING $Q_n(z)$ AROUND THE CUT:



OUTSIDE THE CUT, INTEGRATION CANCELS. ALSO $\oint f_{n-1} = 0$.

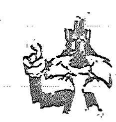
THUS

$$2\oint_C Q_n(z)\,dz = -\int_{-1}^{1}\left[P_n(x)\ln\left(\frac{x+1}{x-1}\right) - i\pi\right]dx + \int_{-1}^{1}\left[P_n(x)\ln\left(\frac{x+1}{x-1}\right) + i\pi\right]dx$$

$$= i2\pi\int_{-1}^{1} P_n(t)\,dt$$

THUS:

$$Q_n(z) = \frac{1}{i2\pi}\oint_C \frac{Q_n(t)\,dt}{t-z} = \frac{i\pi}{i2\pi}\int_{-1}^{1}\frac{P_n(t)\,dt}{t-z} = \frac{1}{2}\int_{-1}^{1}\frac{P_n(t)\,dt}{t-z}$$



7-8. $F(x,h) = e^{2hx-h^2} = \sum_n H_n(x)\frac{h^n}{n!}$

a. $\frac{d}{dh}F(x,h) = 2(x-h)e^{2hx-h^2} = 2(x-h)F(x,h)$

$= \sum H_n \frac{n\,h^{n-1}}{n!} = \sum_n H_n \frac{h^{n-1}}{(n-1)!}$

$$\therefore \sum H_n \frac{h^{n-1}}{(n-1)!} = 2(x-h)\sum_m H_m(x)\frac{h^m}{m!}$$

$$= \underbrace{\sum_m 2x\frac{h^m}{m!}H_m(x)}_{m=n-1} - \underbrace{\sum_m \frac{2h^{m+1}}{m!}H_m}_{m=n-2}$$

$$= \sum_n 2x\frac{h^{n-1}}{(n-1)!}H_{n-1}(x) - \sum_n \frac{2}{(n-2)}\frac{h^{n-1}}{(n-1)!}H_{n-2}$$

EQUATING EQUAL COEFFICIENTS OF h

$$\frac{1}{(n-1)!}H_n = \frac{1}{(n-1)!}xH_{n-1} - \frac{1}{(n-1)!}\frac{2}{(n-2)}H_{n-2}$$

$$H_n = xH_{n-1} - \frac{2}{(n-2)}H_{n-2}$$

$$m = n-1 \Rightarrow n = m+1$$

$$H_{m+1} = xH_m - \frac{2}{(m-1)}H_{m-1}$$

$$\Rightarrow xH_m(x) = H_{m+1}(x) + \frac{2}{(m-1)}H_{m-1}(x)$$

b. EVALUATE

$$I_n = \int_{-\infty}^{\infty} H_n(x) e^{-x^2/2} dx$$

$$\sum_{n=0}^{\infty} H_n \frac{h^n}{n!} = e^{2hx-h^2}$$

$$\sum_{n=0}^{\infty} H_n e^{-x^2/2} \frac{h^n}{n!} = e^{+2hx-h^2-x^2/2}$$

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} \int_{-\infty}^{\infty} H_n(x) e^{-x^2/2} dx = e^{-h^2} \int_{-\infty}^{\infty} e^{2hx-\frac{x^2}{2}} dx$$

$$\sum_{n=0}^{\infty} I_n \frac{h^n}{n!} = e^{-h^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-4hx)} dx$$

$$= e^{-h^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-2h)^2-4h^2]} dx$$

$$= e^{-h^2} \int_{-\infty}^{\infty} e^{2h^2} e^{-\frac{1}{2}(x-2h)^2} dx$$

$$= e^{h^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-2h)^2} dx$$

$$= e^{h^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$$

$$= e^{h^2} \sqrt{2\pi}$$

$$= \sqrt{2\pi}\left[1 + h^2 + \frac{h^4}{2!} + \frac{h^6}{3!} + \dots\right]$$

THUS

$$I_0 + I_1 h + I_2 \frac{h^2}{2!} + I_3 \frac{h^3}{3!} + I_4 \frac{h^4}{4!} + \dots$$

$$= \sqrt{2\pi}\left[1 + h^2 + \frac{h^4}{2!} + \frac{h^6}{3!} + \dots\right]$$

EQUATING EQUAL POWERS OF h:

$$I_{2n+1} = \int_{-\infty}^{\infty} H_{2n+1} e^{-x^2/2} dx = 0$$

THIS LEAVES

$$\sum_{n=0}^{\infty} I_{2n} \frac{h^{2n}}{(2n)!} = \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{h^{2n}}{n!}$$

$$\Rightarrow I_{2n} \frac{1}{(2n)!} = \sqrt{2\pi} \frac{1}{n!}$$

$$I_{2n} = \sqrt{2\pi} \frac{(2n)!}{n!}$$

THUS:

$$\int_{-\infty}^{\infty} H_n(x) e^{-x^2/2} dx = \begin{cases} \sqrt{2\pi} \frac{n!}{(n/2)!} & ; n \text{ EVEN} \\ 0 & ; n \text{ ODD} \end{cases}$$

2-9. ① $f_0(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$

② $(n+1) f_{n+1} = x f_n - f_{n+2} \Rightarrow n f_n = x f_{n-1} - f_{n+1}$

③ $f_n' = f_{n-1}$

FIND $G(x,t) = \sum_{n=-\infty}^{\infty} f_n(x)\, t^n$

$$\frac{d}{dx} G(x,t) = \sum_n f_n' t^n$$
$$= \sum_n f_{n-1} t^n \quad \leftarrow \text{FROM ③}$$
$$= t \sum_n f_{n-1} t^{n-1}$$
$$= t\, G(x,t)$$

$$\frac{d}{dt} G(x,t) = \sum f_n\, n\, t^{n-1}$$
$$= \sum_n \left[x f_{n-1} - f_{n+1}\right] t^{n-1} \quad \longleftarrow \text{FROM ②}$$
$$= \sum_n x f_{n-1} t^{n-1} - \sum_n f_{n+1} t^{n-1}$$
$$= x \sum_n f_n t^n - \frac{1}{t^2} \sum_n f_{n+1} t^{n+1}$$
$$= x\, G(x,t) - \frac{1}{t^2} G(x,t)$$
$$= \left(x - \frac{1}{t^2}\right) G(x,t)$$

A SOLUTION TO THESE TWO PARTIAL DIFFER. EQUAS IS

$$G(x,h) = C e^{xt + 1/t} \quad \ni C = \text{CONSTANT}$$

NOW WE GOTTA TRY IT OUT:

$$\sum_n f_n t^n = G(x,t) = C e^{xh + 1/t} = C e^{xh} e^{1/t}$$
$$= C \left[\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right]\left[\sum_{m=0}^{\infty} \frac{t^{-m}}{m!}\right]$$
$$= C\left[1 + \frac{xt}{1!} + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + \frac{(xt)^4}{4!} + \frac{(xt)^5}{5!} + \dots\right]$$
$$\times \left[1 + \frac{t^{-1}}{1!} + \frac{t^{-2}}{2!} + \frac{t^{-3}}{3!} + \frac{t^{-4}}{4!} + \frac{t^{-5}}{5!} + \dots\right]$$

COLLECTING TERMS WITH SAME POWER OF t:

$$G(x,t) = \dots + C\left[1 + \frac{x}{2!} + \frac{x^2}{2!\,3!} + \frac{x^3}{3!\,4!} + \frac{x^4}{4!\,5!} + \dots\right] t^{-1}$$
$$+ C\left[1 + x + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} + \dots\right] t^0$$
$$+ C\left[\frac{x^2}{2!} + \frac{x^3}{1!\,3!} + \frac{x^4}{2!\,4!} + \frac{x^5}{3!\,5!} + \dots\right] t + \dots$$

RECOGNIZING COEFFICIENT OF t^n IS f_n GIVES,

FROM ①, $C = 1$, AND $G(x,t) = e^{xt + 1/t}$

7-10. $x^2 y'' + x y' + (x^2 - m^2) y = 0$

LET g AND f BE IND. SOLUTIONS OF BESSEL'S EQN. FOR A GIVEN m

$$\Rightarrow \begin{cases} x^2 g'' + x g' + (x^2 - m^2) g = 0 \\ x^2 f'' + x f' + (x^2 - m^2) f = 0 \end{cases}$$

$$\begin{cases} x^2 f g'' + x f g' + (x^2 - m^2) f g = 0 \\ x^2 g f'' + x g f' + (x^2 - m^2) f g = 0 \end{cases}$$

SUBTRACTING GIVES

$$x^2 [f g'' - g f''] + x (f g' - g f') = 0$$

$$x [f g'' - g f''] + (f g' - g f') = 0$$

NOW:

$$W[f, g] = f g' - g f'$$

$$\frac{d}{dx} W[f, g] = (f' g' - g' f') + (f g'' - g f'')$$

$$= f g'' - g f''$$

THUS

$$x W' + W = 0$$

$$\frac{dW}{W} + \frac{dx}{x} = 0$$

$$\ln W + \ln x = C' \quad \Rightarrow C' = \text{CONSTANT}$$

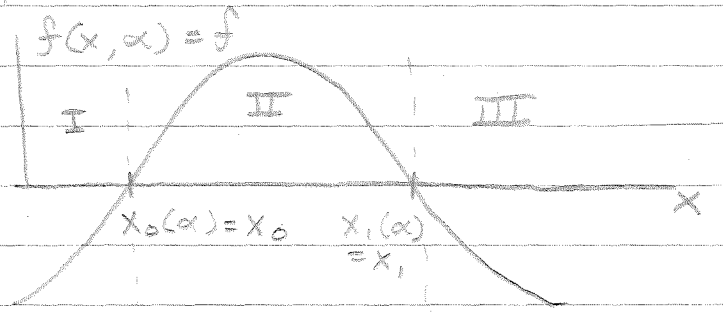
$$\ln W = \ln C/x$$

$$W = \frac{C}{x} = f g' - g f'$$

BOB MARKS
10-9-75

FOR A FUNCTION $f(x;\alpha)$ PARAMETERIZED IN α, WHAT CONDITIONS MUST BE FULFILLED SUCH THAT Y, A SOLUTION OF $Y'' + f(x;\alpha)Y = 0$, IS BOUNDED.

ROUGHLY



THE BOUNDED BOUNDARY CONNECTION RELATIONSHIP FOR REGIONS I AND II IS

$$\frac{A}{\sqrt[4]{-f}}\, e^{-\int_x^{x_0}\sqrt{-f}\,dx} \longrightarrow \frac{2A}{\sqrt[4]{f}} \cos\left[\int_{x_0}^{x}\sqrt{f}\,dx - \frac{\pi}{4}\right]$$

AND FOR REGIONS II AND III:

$$\frac{2A}{\sqrt[4]{f}} \cos\left[\int_x^{x_1}\sqrt{f}\,dx - \frac{\pi}{4}\right] \longleftarrow \frac{B}{\sqrt[4]{-f}}\, e^{\int_{x_1}^{x}\sqrt{-f}\,dx}$$

Y IS THUS BOUNDED WHEN

$$\cos\left[\int_{x_0}^{x}\sqrt{f}\,dx - \frac{\pi}{4}\right] = \pm\cos\left[\int_{x_1}^{x}\sqrt{f}\,dx + \frac{\pi}{4}\right]$$

OR EQUIVALENTLY WHEN

$$\int_{x_0}^{x}\sqrt{f}\,dx = \int_{x_1}^{x}\sqrt{f}\,dx + \pi n + \frac{\pi}{2} \ni n \in \text{INT.}$$

$$\int_{x_0}^{x}\sqrt{f}\,dx + \int_{x}^{x_1}\sqrt{f}\,dx = (n+\tfrac{1}{2})\pi$$

$$\int_{x_0}^{x_1}\sqrt{f}\,dx = (n+\tfrac{1}{2})\pi$$

THUS, Y, FOR $f(x,\alpha)$ A SKETCHED ABOVE, IS BOUNDED IF

$$\int_{x_0(\alpha)}^{x_1(\alpha)}\sqrt{f(x,\alpha)}\,dx = (n+\tfrac{1}{2})\pi$$

(1) [illegible] general second order linear differential equation for $y(z)$ is

$$y'' + P(z)\,y' + Q(z)\,y = 0 \qquad \text{(homogeneous)}$$

[illegible] (7-13) p.198. For this problem two facts will be extremely useful to you:

(i) the footnote on p.188 ([illegible] analytic function);

(ii) the conditions for the point at ∞ to be a regular point or a regular singular point as given at the top of p.210 [NOTE: a function is "regular" at ∞ if it is "finite" at ∞.]

(2) Find the most general solution of the most general second order homogeneous differential equation with:

(a) no singular points at all (even at ∞);

(b) exactly one ~~singular~~ regular singular point (at $z = a$);

(c) exactly one regular singular point (at $z = \infty$).

(3) Prove or give a solid argument to show that an analytic function with no singularities in the finite plane and without an essential singularity at ∞ must be a polynomial.

[1] POWER SERIES EXPANSION

Consider the differential equation

$$y'' + \frac{z^2+3}{16 z^2}\, y = 0 \qquad \text{for } y(z) \text{ where } z \text{ is dimensionless}$$

Make a "power" series expansion for $y(z)$ about the point $z = 0$:

(a) Identify every singular point of this DE and classify each as a regular singular point or an essential singular point.

(b) Find the general recursion relations for the coefficients in each of two linearly independent solutions. [Do not solve the recursion relations for closed form expressions for the A_n's]

(c) By considering the behavior of your solutions near the origin argue that they are, in fact, linearly independent.

(d) For each solution show the first three non-zero terms explicitly [do numerical multiplications, but not divisions].

(e) Any way and as best you can, discuss the nature of your general solution as $z \to \infty$.

[illegible] PARTIAL FRACTION EXPANSIONS

MITOG-LFFLEP
53

It's possible to make expansions of the style:

$$f(z) = \frac{1}{(z-a)(z-b)(z-c)} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c}$$

and a well known trick allows the evaluation of the coefficients A, B, C very easily from the formula

$$f(z) = \sum_n \frac{R_n}{z-a_n}$$

where the a_n are the first order poles of $f(z)$ at a, b, c and the R_n are the residues of $f(z)$ at the poles.

(a) Using the closed contour integral:

$$\oint_C \frac{f(z')}{z'-z}\,dz' \quad \text{for suitable contours } C$$

1st order poles

derive the procedure above for functions with nothing worse than a finite number of first order poles. Either get a formula that always works or tell for what types of functions your formula fails.

(b) Extend your procedure to an infinite number of poles by applying it to

$$f(z) = \frac{1}{\sin \pi z} = \text{(give the series)}$$

Prove or argue convincingly that the result is right; i.e. that it is okay to have infinitely many poles in an [illegible]

[IV] Define a function $\sqrt{z} \equiv f(z)$ such that $(\sqrt{z})^2 = z$ with one branch point and three separate branch cuts going all the way, continuously, to infinity. Make your definition clean and precise. Evaluate the discontinuity across each of your cuts, explaining clearly what you mean by "the" discontinuity. Then perform the contour integral

a sign difference

$$\oint f(z)\,dz$$ around a unit circle centered at the branch point.

[V] Work PROBLEM (7-20) on page 213.

RJ M MARKS
BOB MARKS

I. $Y'' + P(z)Y' + Q(z)Y = 0$

a. No singular points $\rightarrow P(z), Q(z)$ are analytic. NO!

THUS $\rightarrow$ Y is analytic

THUS $\rightarrow$ Y may be expressed as a Taylor series:

$$Y = \Sigma A_n (z - z_0)^n$$

where z_0 is the arbitrary regular point about which the expansion takes place. Since the Taylor series expansion is unique for a given Y (with the two arbitrary constants specified), Y may be expressed in a Mac Laurin series (ie $z_0 = 0$)

$$Y = \sum_{n=0}^{\infty} A_n z^n$$

IMPOSSIBLE! there is no such equation.

Will converge for all z

RJM

The most general such eqn is $y'' + 2y' = 0 \Rightarrow y = Ae^{-2z} + B$

b. One singular point @ $z = a$

$$Y'' + \frac{P_R(z)}{z-a} Y' + \frac{Q_R(z)}{(z-a)^2} Y = 0$$

$P_R(z)$ & $Q_R(z)$ are regular.

Solution here is

$$Y = (z-a)^{\sigma} \sum_{n=0}^{\infty} A_n (z-a)^n$$

Here, the trouble can be extracted by "factoring" out a $(z-a)^{\sigma}$ from the taylor series. ~~Again~~, σ will be determined by the ~~as~~ P_R & Q_R. ~~@~~ Two A_n's are arbitrary constants.

Y will converge for all z.

The eqn is $y'' = 0 \Rightarrow y = Az + B$

c. D.E. Would look like

$$Y'' + (z-a) P_R(z) + (z-a)^2 Q_z(z) = 0$$

where both P_R & Q_z are regular everywhere

RJM

d. As stated before, all analytic functions can be expressed as a unique Talor series, and thus, as a unique (infinate order/converging) polynomial.

That's not what "polynomial" means!

By failing to force the desired behavior at ∞, you missed the whole point of the problem.

I

W

RJM

II. $Y'' + \frac{z^2+3}{16 z^2} Y = 0$

a. THE Y' TERM COEFFICIENT is missing. This coefficient thus has a regular singular point @ $|z| = \infty$. The Y coefficient has a double order pole @ $z=0$. This is also a regular singular point.

We note then, that this differential equation ~~has the maximum number of essential singularities (at the~~ ~~st~~ may be solved around $z=0$ using

$$Y = z^{\sigma} \sum_{n=0}^{\infty} A_n z^n \quad ; \; A_0 \neq 0$$

∞ is essential singularity!

RJM

b. $Y'' + \frac{Z^2+3}{16Z^2} Y = 0$

LET $Y = Z^\sigma \sum_{n=0}^{\infty} A_n Z^n \Rightarrow A_0 \neq 0$

$Y'' + \frac{1}{16} Y + \frac{3}{16Z^2} Y = 0$

$16Y'' + Y + \frac{3}{Z^2} Y = 0$

$16 \; Y''$	Y	$\frac{1}{Z^2} Y$
$Y = A_{n+2} Z^{n+2+\sigma}$	$Y = A_n Z^{n+\sigma}$	$Y = A_{n+2} Z^{n+2+\sigma}$
$Y'' = A_{n+2}(n+\sigma+2)(n+\sigma+1) Z^{n+\sigma}$		$\frac{1}{Z^2} Y = A_{n+2} Z^{n+\sigma}$

$\therefore 16 A_{n+2}(n+\sigma+2)(n+\sigma+1) + A_n + 3A_{n+2} = 0$

RIGHT!

$[16(n+\sigma+2)(n+\sigma+1) + 3] A_{n+2} + A_n = 0 \Leftarrow$

Gotta now look at behavior for "small" n. ~~Assume that A_0 and A_1 are inputs~~. Check

$n = $ "-1" gives: ~~Y~~ ~~$n=0$~~

$16 A_1(\sigma+1)\sigma + 3A_1 = 0$

$16(\sigma+1)\sigma = -3$

$(\sigma+1)\sigma + 3/16 = 0$

$\sigma^2 + \sigma + 3/16 = 0$

GIVES

n = 0 is the important point

$$\sigma = \frac{-1 \pm \sqrt{1 - \frac{12}{16}}}{2}$$

$$= \frac{-1 \pm \sqrt{4/16}}{2} = \frac{-1 \pm \frac{1}{2}}{2} = \frac{-2 \pm 1}{4}$$

$$= -\frac{3}{4}, -\frac{1}{4}$$

RJM

c. The two arbitrary constants may be A_0 & A_1. A_0 generates even terms and A_1 gives the odd terms. ~~In one case to~~ Without loss of generality, assume A_0 & A_1 are ~~not~~ the same for both solutions (both 0). Then void of the z^0 term, both are the same taylor series. One solution is multiplied by $z^{-3/4}$ and the other by $z^{-1/4}$. Obviously, they are independent.

Must have $A_0 \neq 0$ $A_1 = 0$

RJM

Given A_0 & A_1

$\textcircled{0}$

$$A_2 = \frac{-A_0}{16(\sigma+2)(\sigma+1)+3}$$

$$A_3 = \frac{-A_1}{16(3+\sigma)(2+\sigma)+3}$$

FOR $\sigma = -\frac{1}{4}$

$$A_2 = \frac{-A_0}{16(7/4)(3/4)+3}$$

$$= \frac{-A_0}{24}$$

$$A_3 = \frac{-A_1}{16(\frac{11}{4})(\frac{7}{4})+3}$$

$$= \frac{-A_1}{80}$$

FOR $\sigma = -3/4$

$$A_2 = \frac{-A_0}{16(5/4)(\frac{1}{4})+3}$$

$$= -A_0/8$$

$$A_3 = \frac{-A_1}{16(9/4)(5/4)+3}$$

$$= \frac{-A_1}{48}$$

ETC.

RJM

e. @ $z=\infty$, we get something like

$$Y'' + \frac{1}{16} Y = 0$$

$$\Rightarrow Y = c_1 \sin \frac{x}{4} + c_2 \cos \frac{x}{4}$$

ie solution should look sinusoidal for large x (ie, obey like sinusoids in the complex plane for large z.)

GOOD!

II

LB

RJM

not if $f(z)$ has poles inside C

III. a. $f(z) = \frac{1}{i2\pi} \oint_C \frac{f(z')}{z'-z}\, dz'$

$$f(z) = \frac{f_R(z)}{\prod_{i=1}^{m} (z - z_i)} \Leftarrow m \text{ POLES}$$

assume $z_i \neq z_j \ \forall\, i \neq j$
(ie simple 1^{ST} order poles)
$f_R(z)$ is regular

Then

$$\oint_C f(z)\,dz = \oint_C \frac{f_R(z)}{\prod_{i=1}^{m}(z - z_i)}$$

What is $f_R(z)$?

Define C_n such that it encloses only the pole @ z_n

I suggested that you use $\oint \frac{f}{z'-z} dz'$ instead of $\oint f\, dz'$

$$\oint_{C_n} f(z)\,dz = \oint_{C_n} \frac{f_R(z)}{(z - z_n) \prod_{\substack{i=1 \\ n \neq m}}^{m} (z - z_i)} dz$$

Within C_n, $\frac{f_R(z)}{\prod_{\substack{i=1 \\ n \neq m}}^{m} (z - z_i)}$ is analytic. Thus, by Cauchy's theorem

$$\frac{1}{i2\pi} \oint_{C_n} \frac{1}{z - z_n} \frac{f_R(z)}{\prod_{\substack{i=1 \\ n \neq m}}^{m} (z - z_i)} = \frac{f_R(z_n)}{\prod_{\substack{i=1 \\ n \neq m}}^{m} (z_n - z_i)}$$

$$= f(z)(z - z_n)\Big|_{z=z_n} \equiv \operatorname{Res} f(z)\Big|_{z=z_n}$$

RJM

NOW

$$f(z) = \frac{f_R(z)}{\prod_{i=1}^{m}(z - z_i)} =$$

$$= \sum_{i=1}^{m} \frac{R_i}{z - z_i}$$

$$\oint_{C_n} f(z)\,dz = \sum \oint_{C_n} \frac{R_i}{z - z_i}$$

$$= \oint_{C_n} \frac{R_n}{z - z_n}$$

Not really a derivation; more of a verification. Clumsy, hard to generalize.

THUS $R_n = \text{Res}\, f(z)\big|_{z=z_n}$

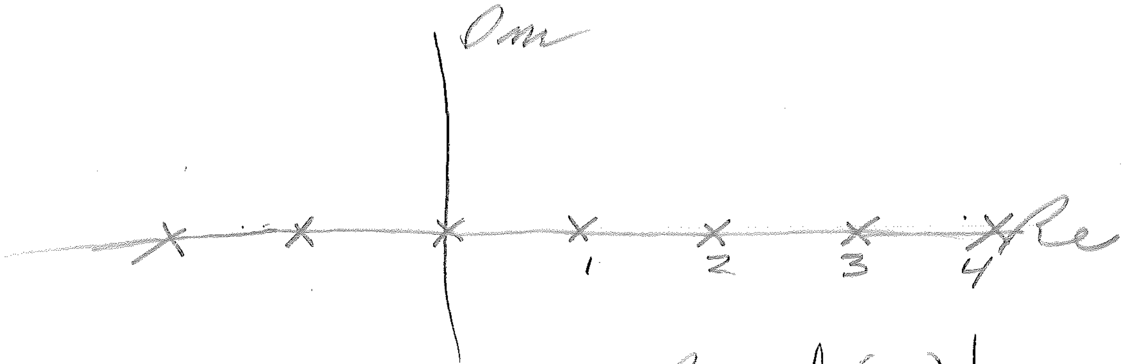
Thus, for functions with m single order poles and analytic elsewhere, we have the identity

$$f(z) = \sum_{i=1}^{m} \frac{R_n}{z - z_i} \Rightarrow R_n = \text{Res}\, f(z)\big|_{z=z_n}$$

QED

RJM

b. $f(z) = \dfrac{1}{\sin \pi z}$ $\quad m \to \infty$



$$f(z) = \sum_{p=-\infty}^{\infty} \frac{\operatorname{Res} f(z)\big|_{z=z_p}}{z - z_p}$$

$z_p = p = \ldots -2, -1, 0, 1, 2 \ldots$

$$f(z) = \sum_{p=-\infty}^{\infty} \frac{\operatorname{Res} f(z)\big|_{z=p}}{z - p}$$

$$R_p = \operatorname{Res} f(z)\big|_{z=p} = f(z)(z-p)\big|_{z=p}$$

$$= \frac{z-p}{\sin \pi z}\bigg|_{z=p}$$

use LA HOPITAL:

$$R_p = \lim_{z \to p} \frac{z-p}{\sin \pi z}$$

$$= \lim_{z \to p} \frac{1}{\pi \cos \pi z} = \frac{1}{\pi}(-1)^p$$

THUS

$$\frac{1}{\sin \pi z} = \frac{1}{\pi} \sum_{p=-\infty}^{\infty} \frac{(-1)^p}{z - p}$$

RJM

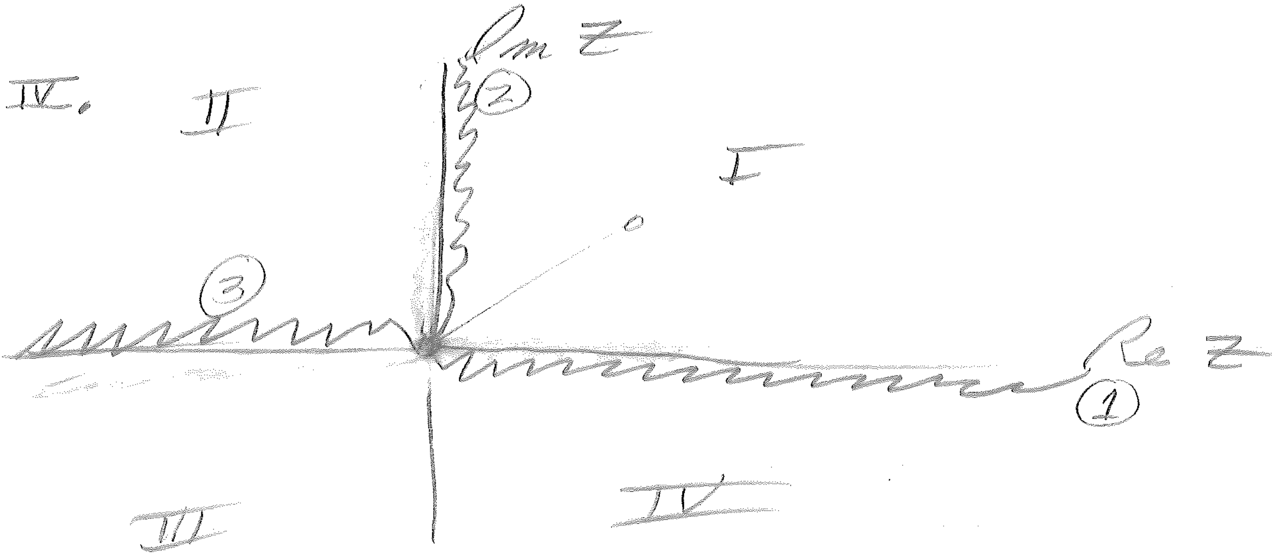
The answer looks good - It is periodic for Re(z) with period 2P and has poles in the right places, and changes signs in the right places.

Good, takes cleverness to see this check!

In general, it would seem that if in ~~any~~ every finite region of the complex plane there existed at most a finite number of first order poles, ~~then~~ we could let $m \to \infty$. ~~For~~ The method could obviously be extended to higher order poles. Essential singularities, though, would probably not be applicable. Something like $\sin(1/z^2)$ would intuitively ~~not to~~ seem not to work here.

III LB

IV.



Branch point @ origin

For z in quadrant I: $\left(z = re^{i2\pi\phi}\right)$ $e^{i2\pi n}$

$$\sqrt{z} \triangleq \sqrt{r_1}\, e^{i\theta_1/2}$$

$$(\sqrt{z})^2 = re^{i\theta} = z$$

$$0 \le \theta_1 < \pi/2$$

For z in quadrant II

But how will you measure θ_2?

$$\sqrt{z} \triangleq \sqrt{r_2}\, e^{i\left(\frac{\theta_2}{2}+\pi\right)}$$

$$(\sqrt{z})^2 = r_2 e^{i(\theta_2+2\pi)} = z e^{i2\pi} = z$$

$$\frac{\pi}{2} \le \theta_2 < \pi$$

For z in III or IV

Doesn't do anything

$$\sqrt{z} = \sqrt{r_3}\, e^{i\left(\frac{\theta_3}{2} - 2\pi\right)}$$

$$\pi \le \theta_3 < 2\pi$$

all θ measured CC from positive Re(z) axis

Discontinuity across cuts.

Defn: Discontinuity: Going CC across a cut, the initial value of $\sqrt{z}$ / ~~final (through~~ cut) value of $\sqrt{z}$. $D = 1 \Rightarrow$ NO DISCONTINUITY

$D = -1 \Rightarrow$ SIGN CHANGE

This is a RATIO not a "DISCONTINUITY"

iY

x

①

$\sqrt{z} = \sqrt{x}$

$\sqrt{z} = \sqrt{x}\, e^{-i\pi} = -\sqrt{x}$ $= \cancel{\sqrt{x}\, e^{i\pi}} = -\sqrt{x}$

$D_1 = -1$

iY

2

$\sqrt{z} = \sqrt{Y}\, e^{i\frac{\pi}{4}}$

$\sqrt{z} = \sqrt{Y}\, e^{i\frac{5\pi}{4}} = -\sqrt{Y}\, e^{i\frac{\pi}{4}}$

$D_2 = -1$

$\sqrt{z} = \sqrt{-x}\, e^{i\frac{3\pi}{2}} = -i\sqrt{-x}$

③

$i^{\pi/2}$

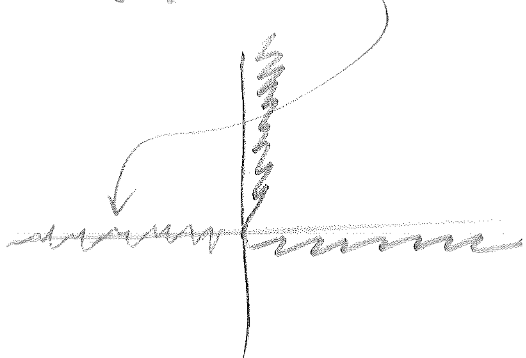
$\sqrt{z} = \sqrt{-x}\, e^{-\frac{5\pi}{2}} = i\sqrt{-x}$ NO

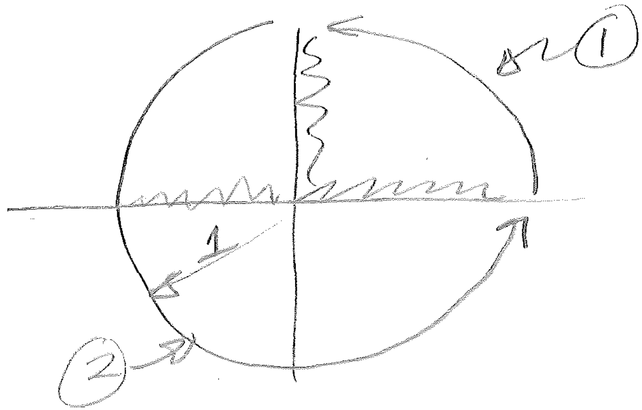
$D_3 = -1$

$\frac{\pi}{2} - \frac{6\pi}{2}$

Looks, off hand, that given 3 linear cuts from 0 to infinity, there is no way to define $\sqrt{z}$ such that at least one cut won't have a discontinuity, since the most you can have across a cut for $\sqrt{z}$ is a sign change. Thus, for three cuts, the most you can have is ~~a~~ two sign changes.

Anyway, this leaves us with the cuts: as a mistake





(1) $\int_{\theta_1=0}^{\pi/2} \sqrt{z}\,dz = \int_0^{\pi/2} e^{i\frac{\theta_1}{2}}\,d\theta$ ✓

$$= \frac{2}{i} e^{i\frac{\theta_1}{2}}\Big|_0^{\pi/2}$$
$$= \frac{2}{i}\left[e^{i\frac{\pi}{4}} - 1\right]$$
$$= \frac{2}{i}\left[\frac{1}{\sqrt{2}}(1+i) - 1\right] = T_2$$

(2) $e^{i\pi}\int_{\theta=\pi/2}^{\pi} e^{i\frac{\theta}{2}} = -\frac{2}{i}\, e^{i\frac{\theta}{2}}\Big]_{\pi/2}^{\pi}$ ✓

$$= -\frac{2}{i}\left[e^{i\pi/2} - e^{i\pi/4}\right]$$
$$= -\frac{2}{i}\left[i - \frac{1}{\sqrt{2}}(1+i)\right] = T_1$$

(3) $\int_{\pi}^{2\pi} e^{i\frac{\theta}{2}} = \frac{2}{i}\, e^{i\theta/2}\Big|_{\pi}^{2\pi}$ ✓

$$= \frac{2}{i}\left[-1 - e^{i\pi/2}\right]$$
$$= \frac{2}{i}\left[-1 - i\right] = T_3$$

$$\oint_C \sqrt{z}\,dz = T_1 + T_2 + T_3$$

FINISH
DO!

IV LB

V. $x^\alpha J_{\pm m}(\beta x^\gamma)$

Well $J_{\pm m}(x)$ is solution to

$$x^2 y'' + x y' + (x^2 - m^2) y = 0$$
$$x^2 J_m'' + x J_m' + (x^2 - m^2) J_m = 0$$

$$\cancel{\frac{d}{dx} x^\alpha J_{\pm m}(\beta x^\gamma)}$$

$$\cancel{= \alpha x^{\alpha-1} J_{\pm m}(\beta x^\gamma) + \beta x^{\alpha-\gamma} J'_{\pm m}(\beta x^\alpha)}$$

$$H(x) = x^\alpha J_\pm(\beta x^\alpha)$$

$$\frac{d}{dx} x^\alpha J_\pm(\beta x^\gamma) = \frac{d}{dx} H$$

$$= \alpha x^{\alpha-1} J_\pm(\beta x^\alpha) + \underline{x^\alpha \beta \gamma x^{\gamma-1} J'_{\pm m}(\beta x^\gamma)}$$

$$\frac{dH}{dx} = \frac{\alpha}{x} x^\alpha J_\pm(\beta x^\alpha) + \overbrace{\qquad\qquad}$$

$$= \frac{\alpha}{x} H + \frac{dH}{dx} \Rightarrow \frac{dH}{dx} = \frac{\alpha H}{x}$$

HMMM...

RJM

V. 7-20

Bessel's Eq is

$$x^2 y'' + x y' + (x^2 - m^2) y = 0$$

$$x^2 J_m'' + x J_m' + (x^2 - m^2) J_m = 0$$

$$H = x^\alpha J_m(\beta x^\gamma)$$

$$\cancel{H' = \alpha x^{\alpha-1} J_m(\beta x^\gamma) + x^\alpha \gamma x^{\gamma-1} J_m'(\beta x^\gamma)}$$

$$H x^{-\alpha} = J_m(\beta x^\gamma)$$

SUBSTITUTING GIVES

$$(\beta x^\gamma)^2 (H x^{-\alpha})'' + (\beta x^\gamma)(H x^{-\alpha})' + ((\beta x^\gamma)^2 - m^2) H x^{-\alpha} = 0$$

$$\Rightarrow (\;)^{(n)} = \left(\frac{d}{d(\beta x^\gamma)}\right)^n = \frac{1}{\beta^n}\left(\frac{d}{d(x^\gamma)}\right)^n$$

$$(x^{2\gamma})(H x^{-\alpha})'' + x^\gamma (H x^{-\alpha})' + ((\beta x^\gamma)^2 - m^2) H x^{-\alpha}$$

$$\Rightarrow (\;)^n = \left(\frac{d}{dx^\gamma}\right)^n$$

Good idea, this would work!

~~NOT WORKIN TO HOT~~ . .

RJM

$$H x^{-\alpha} = J_m(\beta x^{\gamma})$$

$$\Rightarrow x^{2\gamma} \frac{d^2 J_m(\beta x^{\gamma})}{d^2(x^{\gamma})} + x^{\gamma} \frac{d J_m(\beta x^{\gamma})}{d(x^{\gamma})}$$

$$+ \left[(\beta x^{\gamma})^2 - m^2\right] J_m(\beta x^{\gamma}) = 0$$

$$\xi = x^{\gamma} \qquad \cancel{x^{\gamma}}$$

$$\beta^2 \xi^2 J'' + \xi \beta J' + \left[(\beta \xi)^2 - m^2\right] J' = 0$$

HMMMMM....

RJM

$$H = x^{\alpha} J_m(\beta x^{\gamma})$$

$$\frac{dH}{dx} = \alpha x^{\alpha-1} J_m(\beta x^{\gamma}) + x^{\alpha}\beta\gamma x^{\gamma-1} J_m'(\beta x^{\gamma})$$

$$= \frac{\alpha}{x} H + x^{\alpha+\gamma-1}\beta\gamma J_m'(\beta x^{\gamma})$$

$$= \frac{\alpha}{x} H_m + x^{\alpha+\gamma-1}\beta\gamma\left[J_{m-1}(\beta x^{\gamma}) - \frac{m}{\beta x^{\gamma}} J_m(\beta x^{\gamma})\right]$$

$$= \frac{\alpha}{x} H_m + x^{\alpha+\gamma-1}\beta\gamma\left[J_{m-1}(\beta x^{\gamma})\right)$$

$$- x^{\alpha+\gamma-1}\beta\gamma \frac{m}{\beta x^{\gamma}} J_m(\beta x^{\gamma})$$

$$\sum_n J_n(x) h^n = e^{\frac{x}{2}\left(h-\frac{1}{h}\right)}$$

$$\sum_n x^{\alpha} J_m(\beta x^{\gamma}) h^n = x^{\alpha} e^{\frac{1}{2}\beta x^{\gamma}\left(h-\frac{1}{h}\right)}$$

GOING NOWHERE ... –

got off the track a little,
but you were close!

GOTTA IDEA!

$$\underset{\sim}{x}^2 Y'' + \underset{\sim}{x} Y' + (\underset{\sim}{x}^2 - m^2) Y = 0$$

$$\underset{\sim}{x} \rightarrow \beta x^{\gamma}$$

$$\frac{d\tilde{x}}{dx} = \gamma \beta x^{\gamma-1} \Rightarrow \tilde{Y}(\underset{\sim}{x}) = Y(\beta x^{\gamma})$$

$$\frac{dY}{d\underset{\sim}{x}} \frac{d\tilde{x}}{dx} = \frac{dY}{dx}$$

$$\frac{dY}{dx} = \gamma \beta x^{\gamma-1} \frac{dY}{d\underset{\sim}{x}}$$

$$\frac{dY}{d\underset{\sim}{x}} = \frac{x^{1-\alpha}}{\gamma\beta} \frac{dY}{dx}$$

$$\frac{d^2 Y'}{d\underset{\sim}{x}^2} = \frac{d}{d\underset{\sim}{x}} \frac{x^{1-\alpha}}{\gamma\beta} \frac{dY}{dx}$$

$$= \frac{x^{1-\alpha}}{\gamma\beta} \frac{d}{d\underset{\sim}{x}} \frac{dY}{dx} + \frac{dY}{dx} \frac{d}{d\tilde{x}}$$

HMMMMM....

TIMES UP

I QUIT

V

LB